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Spectral features and asymptotic properties for α-circulants and α-Toeplitz sequences: theoretical results and examples

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Abstract

For a given nonnegative integer α, a matrix A_n of size n is called α-Toeplitz if its entries obey the rule $A_n = [a_{r-\alpha s}]_{r,s=0}^{n-1}$. Analogously, a matrix A_n again of size n is called α-circulant if $A_n = \left[a_{(r-\alpha s) \bmod n}\right]_{r,s=0}^{n-1}$. Such kind of matrices arises in wavelet analysis, subdivision algorithms and more generally when dealing with multigrid/multilevel methods for structured matrices and approximations of boundary value problems. In this paper we study the singular values of α-circulants and we provide an asymptotic analysis of the distribution results for the singular values of α-Toeplitz sequences in the case where $\{a_k\}$ can be interpreted as the sequence of Fourier coefficients of an integrable function f over the domain $(-\pi, \pi)$. Some generalizations to the block, multilevel case, amounting to choose f multivariate and matrix valued, are briefly considered.

Keywords: circulants, Toeplitz, α-circulants, α-Toeplitz, spectral distributions, multigrid methods.
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1 Introduction

A matrix A_n of size n is called α-Toeplitz if its entries obey the rule $A_n = [a_{r-\alpha s}]_{r,s=0}^{n-1}$, where α is a nonnegative integer. As an example, if $n = 5$ and $\alpha = 3$ then

$$A_n \equiv T_{n,\alpha} = \begin{bmatrix} a_0 & a_{-3} & a_{-6} & a_{-9} & a_{-12} \\ a_1 & a_{-2} & a_{-5} & a_{-8} & a_{-11} \\ a_2 & a_{-1} & a_{-4} & a_{-7} & a_{-10} \\ a_3 & a_0 & a_{-3} & a_{-6} & a_{-9} \\ a_4 & a_1 & a_{-2} & a_{-5} & a_{-8} \end{bmatrix}.$$

Along the same lines, a matrix A_n of size n is called α-circulant if $A_n = \left[a_{(r-\alpha s) \bmod n}\right]_{r,s=0}^{n-1}$. For instance if $n = 5$ and $\alpha = 3$ then we have

$$A_n \equiv C_{n,\alpha} = \begin{bmatrix} a_0 & a_2 & a_4 & a_1 & a_3 \\ a_1 & a_3 & a_0 & a_2 & a_4 \\ a_2 & a_4 & a_1 & a_3 & a_0 \\ a_3 & a_0 & a_2 & a_4 & a_1 \\ a_4 & a_1 & a_3 & a_0 & a_2 \end{bmatrix}.$$

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Such kind of matrices arises in wavelet analysis [6] and subdivision algorithms or, equivalently, in the associated refinement equations, see [5] and references therein. Furthermore, it is interesting to remind that Gilbert Strang [22] has shown rich connections between dilation equations in the wavelets context and multigrid methods [12, 29], when constructing the restriction/prolongation operators [9, 1] with various boundary conditions. It is worth noticing that the use of different boundary conditions is quite natural when dealing with signal/image restoration problems or differential equations, see [18, 15].

In this paper we address the problem of characterizing the singular values of α-circulants and of providing an asymptotic analysis of the distribution results for the singular values of α-Toeplitz sequences, in the case where the sequence of values $\{a_k\}$, defining the entries of the matrices, can be interpreted as the sequence of Fourier coefficients of an integrable function f over the domain $(-\pi, \pi)$. As a byproduct, we will show interesting relations with the analysis of convergence of multigrid methods given, e.g., in [21, 1]. Finally we generalize the analysis to the block, multilevel case, amounting to choose the symbol f multivariate, i.e., defined on the set $(-\pi, \pi)^d$ for some $d > 1$, and matrix valued, i.e., such that $f(x)$ is a matrix of given size $p \times q$.

The paper is organized as follows. In Section 2 we report useful definitions, well-known results in the standard case of circulants and Toeplitz that is when $\alpha = 1$ (or $\alpha = e$, $e = (1, \ldots, 1)$, in the multilevel setting), and a preliminary analysis of some special cases. Section 3 deals with the singular value analysis of α-circulants while in Section 4 we treat the α-Toeplitz case in an asymptotic setting, and more precisely in the sense of the Weyl spectral distributions. Section 5 is devoted to sketch useful connections with multigrid methods, while in Section 6 we report the generalization of the results when we deal the multilevel block case. Section 7 is aimed to draw conclusions and to indicate future lines of research.

2 General definitions and tools

For any $n \times n$ matrix A with eigenvalues $\lambda_j(A)$, $j = 1, \ldots, n$, and for any $m \times n$ matrix B with singular values $\sigma_j(B)$, $j = 1, \ldots, l$, $l = \min\{m, n\}$, we set

$$\mathrm{Eig}(A) = \{\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\}, \qquad \mathrm{Sgval}(B) = \{\sigma_1(B), \sigma_2(B), \ldots, \sigma_l(B)\}.$$

The matrix B^*B is positive semidefinite, since $x^*(B^*B)x = \|Bx\|_2^2 \geq 0$ for all $x \in \mathbb{C}^n$, with * denoting the transpose conjugate operator. Moreover, it is clear that the eigenvalues $\lambda_1(B^*B) \geq \lambda_2(B^*B) \geq \cdots \geq \lambda_n(B^*B) \geq 0$ are nonnegative and can therefore be written in the form

$$\lambda_j(B^*B) = \sigma_j^2, \tag{1}$$

with $\sigma_j \geq 0$, $j = 1, \ldots, n$. The numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0$, $l = \min\{m, n\}$, are called singular values of B, i.e., $\sigma_j = \sigma_j(B)$ and if $n > l$ then $\lambda_j(B^*B) = 0$, $j = l + 1, \ldots, n$. A more general statement is contained in the singular value decomposition theorem (see e.g. [11]).

Theorem 2.1. *Let B be an arbitrary (complex) $m \times n$ matrix. Then:*

(a) There exists a unitary $m \times m$ matrix U and a unitary $n \times n$ matrix V such that $U^*BV = \Sigma$ is an $m \times n$ "diagonal matrix" of the following form:

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad D := \mathrm{diag}(\sigma_1, \ldots, \sigma_r), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

Here $\sigma_1, \ldots, \sigma_r$ are the nonvanishing singular values of B, and r is the rank of B.

(b) *The nonvanishing singular values of B^* are also precisely the number $\sigma_1, \ldots, \sigma_r$. The decomposition $B = U\Sigma V^*$ is called "the singular value decomposition of B".*

For any function F defined on $\mathbb{R}_0^+$ and for any $m \times n$ matrix A, the symbol $\Sigma_\sigma(F, A)$ stands for the mean

$$\Sigma_\sigma(F, A) := \frac{1}{\min\{n, m\}} \sum_{j=1}^{\min\{n,m\}} F(\sigma_j(A)) = \frac{1}{\min\{n, m\}} \sum_{\sigma \in \mathrm{Sgval}(A)} F(\sigma). \tag{2}$$

Throughout this paper we speak also of *matrix sequences* as sequences $\{A_k\}$ where A_k is an $n(k) \times m(k)$ matrix with $\min\{n(k), m(k)\} \to \infty$ as $k \to \infty$. When $n(k) = m(k)$ that is all the involved matrices are square, and this will occur often in the paper, we will not need the extra parameter k and we will consider simply matrix sequences of the form $\{A_n\}$.

Concerning the case of matrix-sequences an important notion is that of spectral distribution in the eigenvalue or singular value sense, linking the collective behavior of the eigenvalues or singular values of all the matrices in the sequence to a given function (or to a measure). The notion goes back to Weyl and has been investigated by many authors in the Toeplitz and Locally Toeplitz context (see the book by Böttcher and Silbermann [4] where many classical results by the authors, Szegö, Avram, Parter, Widom Tyrtyshnikov, and many other can be found, and more recent results in [10, 13, 23, 28, 26, 27]). Here we report the definition of spectral distribution only in the singular value sense since our analysis is devoted to singular values. The case of eigenvalues will be the subject of future investigations.

Definition 2.1. Let $\mathcal{C}_0(\mathbb{R}_0^+)$ be the set of continuous functions with bounded support defined over the nonnegative real numbers, d a positive integer, and θ a complex-valued measurable function defined on a set $G \subset \mathbb{R}^d$ of finite and positive Lebesgue measure $\mu(G)$. Here G will be often equal to $(-\pi, \pi)^d$ so that $e^{i\overline{G}} = \mathbb{T}^d$ with $\mathbb{T}$ denoting the complex unit circle. A matrix sequence $\{A_k\}$ is said to be *distributed* (*in the sense of the singular values*) *as the pair* (θ, G), or to *have the distribution function* θ ($\{A_k\} \sim_\sigma (\theta, G)$), if, $\forall F \in \mathcal{C}_0(\mathbb{R}_0^+)$, the following limit relation holds

$$\lim_{k \to \infty} \Sigma_\sigma(F, A_k) = \frac{1}{\mu(G)} \int_G F(|\theta(t)|)\, dt, \qquad t = (t_1, \ldots, t_d). \tag{3}$$

When considering θ taking values in $\mathcal{M}_{pq}$, where $\mathcal{M}_{pq}$ is the space of $p \times q$ matrices with complex entries and a function is considered to be measurable if and only if the component functions are, we say that $\{A_k\} \sim_\sigma (\theta, G)$ when for every $F \in \mathcal{C}_0(\mathbb{R}_0^+)$ we have

$$\lim_{k \to \infty} \Sigma_\sigma(F, A_k) = \frac{1}{\mu(G)} \int_G \frac{\sum_{j=1}^{\min\{p,q\}} (F(\sigma_j(\theta(t))))}{\min\{p, q\}}\, dt, \qquad t = (t_1, \ldots, t_d),$$

with $\sigma_j(\theta(t)) = \sqrt{\lambda_j(\theta(t)\theta^*(t))} = \lambda_j(\sqrt{\theta(t)\theta^*(t)})$. Finally we say that two sequences $\{A_k\}$ and $\{B_k\}$ are *equally distributed* in the sense of singular values (σ) if, $\forall F \in \mathcal{C}_0(\mathbb{R}_0^+)$, we have

$$\lim_{k \to \infty} [\Sigma_\sigma(F, B_k) - \Sigma_\sigma(F, A_k)] = 0.$$

Here we are interested in explicit formulae for the singular values of α-circulants and in distribution results for α-Toeplitz sequences. In the latter case, following what is known in the standard case of $\alpha = 1$ (or $\alpha = e$ in the multilevel setting), we need to link the coefficients of the α-Toeplitz sequence to a certain symbol.

Let f be a Lebesgue integrable function defined on $(-\pi,\pi)^d$ and taking values in $\mathcal{M}_{pq}$, for given positive integers p and q. Then, for d-indices $r=(r_1,\ldots,r_d), j=(j_1,\ldots,j_d), n=(n_1,\ldots,n_d)$, $e=(1,\ldots,1)$, $\underline{0}=(0,\ldots,0)$, the Toeplitz matrix $T_n(f)$ of size $p\hat{n}\times q\hat{n}$, $\hat{n}=n_1\cdot n_2\cdots n_d$, is defined as follows

$$T_n(f)=[\tilde{f}_{r-j}]_{r,j=\underline{0}}^{n-e},$$

where $\tilde{f}_k$ are the Fourier coefficients of f defined by equation

$$\tilde{f}_j=\tilde{f}_{(j_1,\ldots,j_d)}(f)=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} f(t_1,\ldots,t_d)e^{-i(j_1t_1+\cdots+j_dt_d)}\,dt_1\cdots dt_d, \qquad i^2=-1, \tag{4}$$

for integers j_ℓ such that $-\infty<j_\ell<\infty$ for $1\le\ell\le d$. Since f is a matrix-valued function of d variables whose component functions are all integrable, then the $(j_1,\ldots,j_d)$-th Fourier coefficient is considered to be the matrix whose (u,v)-th entry is the $(j_1,\ldots,j_d)$-th Fourier coefficient of the function $(f(t_1,\ldots,t_d))_{u,v}$.

According to this multi-index block notation we can define general multi-level block α-Toeplitz and α-circulants. Of course, in this multidimensional setting, α denotes a d-dimensional vector of nonnegative integers that is $\alpha=(\alpha_1,\ldots,\alpha_d)$. In that case $A_n=[a_{r-\alpha\circ s}]_{r,s=\underline{0}}^{n-e}$ where the $\circ$ operation is the componentwise Hadamard product between vectors or matrices of the same size. A matrix A_n of size $p\hat{n}\times q\hat{n}$ is called α-circulant if $A_n=\left[a_{(r-\alpha\circ s) \bmod n}\right]_{r,s=\underline{0}}^{n-e}$, where

$$(r-\alpha\circ s) \bmod n=((r_1-\alpha_1s_1) \bmod n_1,(r_2-\alpha_2s_2) \bmod n_2,\ldots,(r_d-\alpha_ds_d) \bmod n_d)\,.$$

2.1 The extremal cases where $\alpha=\underline{0}$ or $\alpha=e$, and the intermediate cases

We consider a d-level setting and we analyze in detail the case where $\underline{0}\le\alpha\le e$ and with $\le$ denoting the componentwise partial ordering between real vectors. When α has at least a zero component, the analysis can be reduced to the positive one as studied in Subsection 2.1.3.

2.1.1 $\alpha=e$

In the literature the only case deeply studied is the case of $\alpha=e$ (standard shift in every level). Here for multilevel block circulants $A_n=[a_{(r-\alpha\circ s) \bmod n}]_{r,s=\underline{0}}^{n-e}$ the singular values are given by those of

$$\sigma_k(A_n)=\sum_{j=\underline{0}}^{n-e}a_je^{i2\pi(j_1k_1/n_1+\cdots+j_dk_d/n_d)}, \qquad k=(k_1,\ldots,k_d),$$

for any k_ℓ such that $0\le k_\ell\le n_\ell-1$, $\ell=1,\ldots,d$. Of course when the coefficients a_j comes from the Fourier coefficients of a given Lebesgue integrable function f, i.e. $\tilde{f}_j=a_{j \bmod n}$, $j=-n/2,\ldots,n/2$ $(n/2=(n_1/2,n_2/2,\ldots,n_d/2))$, the singular values are those of $n/2$-th Fourier sum of f evaluated at the grid points

$$2\pi k/n=2\pi\left(k_1/n_1,\ldots,k_d/n_d\right),$$

$0\le k_j\le n_j-1$, $j=1,\ldots,d$. Moreover the explicit Schur decomposition is known. For $d=p=q=1$ any standard circulant matrix can be written in the form

$$A_n\equiv C_n=F_nD_nF_n^*, \tag{5}$$

where

$$\begin{aligned} F_n &= \frac{1}{\sqrt{n}}\left[e^{-\frac{2\pi i jk}{n}}\right]_{j,k=0}^{n-1}, \text{ Fourier matrix,} \\ D_n &= \mathrm{diag}(\sqrt{n}F_n^*\underline{a}), \\ \underline{a} &= [a_0, a_1, \ldots, a_{n-1}]^T, \quad \text{first column of the matrix } A_n. \end{aligned} \tag{6}$$

Of course for general d, p, q the formula generalizes as

$$A_n = (F_n \otimes I_p)D_n(F_n^* \otimes I_q),$$

with $F_n = F_{n_1} \otimes F_{n_2} \otimes \cdots \otimes F_{n_d}$ $D_n = \mathrm{diag}(\sqrt{\hat{n}}(F_n^* \otimes I_p)\underline{a})$, where $\hat{n} = n_1 \cdot n_2 \cdots n_d$ and $\underline{a}$ being the first "column" of A_n whose entries a_j, $j = (j_1, \ldots, j_d)$, ordered lexicographically, are blocks of size $p \times q$.

For multilevel block Toeplitz sequences $\{T_n(f)\}$ generated by an integrable d variate and matrix valued symbol f the singular values are not explicitly known but we know the distribution in the sense of Definition 2.1; see [26]. More precisely we have

$$\{T_n(f)\} \sim_\sigma (f, Q^d), \qquad Q = (-\pi, \pi). \tag{7}$$

2.1.2 $\alpha = \underline{0}$

The other extreme is represented by the case where α is the zero vector. Here the multilevel block α-circulant and α-Toeplitz coincide when $\alpha = \underline{0}$ and are both given by

$$A_n = [a_{(r-\underline{0}\circ s) \bmod n}]_{r,s=\underline{0}}^{n-e} = [a_{r \bmod n}]_{r,s=\underline{0}}^{n-e} = [a_r]_{r,s=\underline{0}}^{n-e} = \begin{bmatrix} a_{\underline{0}} & \cdots & a_{\underline{0}} \\ \vdots & & \vdots \\ a_{n-e} & \cdots & a_{n-e} \end{bmatrix}.$$

A simple computation shows that all the singular values are zero except for few of them given by $\sqrt{\hat{n}}\sigma$, where $\hat{n} = n_1 \cdot n_2 \cdots n_d$ and σ is any singular value of the matrix $(\sum_{j=\underline{0}}^{n-e} a_j^* a_j)^{1/2}$. Of course in the scalar case where $p = q = 1$ the choice of σ is unique and by the above formula it coincides with the Euclidean norm of the first column $\underline{a}$ of the original matrix. In that case it is evident that

$$\{A_n\} \sim_\sigma (0, G),$$

for any domain G satisfying the requirements of Definition 2.1.

2.1.3 When some of the entries of α vanish

The content of this subsection reduces to the following remark: the case of a nonnegative α can be reduced to the case of a positive vector so that we are motivated to treat in detail the latter in the next section. Let α be a d-dimensional vector of nonnegative integers and let $\mathcal{N} \subset \{1, \ldots, d\}$ be the set of indices such that $j \in \mathcal{N}$ if and only if $\alpha_j = 0$. Assume that $\mathcal{N}$ is nonempty, let $t \geq 1$ be its cardinality and $d^+ = d - t$. Then a simple calculation shows that the singular values of the corresponding α-circulant matrix $A_n = [a_{(r-\alpha\circ s) \bmod n}]_{r,s=\underline{0}}^{n-e}$ are zero except for few of them given by $\sqrt{\hat{n}[0]}\sigma$ where

$$\hat{n}[0] = \prod_{j \in \mathcal{N}} n_j, \qquad n[0] = (n_{j_1}, \ldots, n_{j_t}), \quad \mathcal{N} = \{j_1, \ldots, j_t\},$$

and σ is any singular value of the matrix

$$\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j\right)^{1/2}. \tag{8}$$

Here C_j is a d^+-level α^+-circulant matrix with $\alpha^+ = (\alpha_{k_1}, \ldots, \alpha_{k_{d^+}})$ and of partial sizes $n[>0] = (n_{k_1}, \ldots, n_{k_{d^+}})$, $\mathcal{N}^C = \{k_1, \ldots, k_{d^+}\}$, and whose expression is

$$C_j = \left[a_{(r-\alpha\circ s) \bmod n}\right]_{r',s'=\underline{0}}^{n[>0]-e},$$

where $(r - \alpha \circ s)_k = j_k$ for $\alpha_k = 0$ and $r'_i = r_{k_i}$, $s'_i = s_{k_i}$, $i = 1, \ldots, d^+$. Taking into account the above notation, for the α-Toeplitz $A_n = [a_{r-\alpha\circ s}]_{r,s=\underline{0}}^{n-e}$ the same computation shows that all the singular values are zero except for few of them given by $\sqrt{\hat{n}[0]}\sigma$ where σ is any singular value of the matrix

$$\left(\sum_{j=\underline{0}}^{n[0]-e} T_j^* T_j\right)^{1/2}. \tag{9}$$

Here T_j is a d^+-level α^+-Toeplitz matrix with $\alpha^+ = (\alpha_{k_1}, \ldots, \alpha_{k_{d^+}})$ and of partial sizes $n[>0] = (n_{k_1}, \ldots, n_{k_{d^+}})$, $\mathcal{N}^C = \{k_1, \ldots, k_{d^+}\}$, and whose expression is

$$T_j = \left[a_{(r-\alpha\circ s)}\right]_{r',s'=\underline{0}}^{n[>0]-e},$$

where $(r - \alpha \circ s)_k = j_k$ for $\alpha_k = 0$ and $r'_i = r_{k_i}$, $s'_i = s_{k_i}$, $i = 1, \ldots, d^+$. Also in this case, since most of the singular values are identically zero, we infer that

$$\{A_n\} \sim_\sigma (0, G),$$

for any domain G satisfying the requirements of Definition 2.1.

3 Singular values of α-circulant matrices

Of course the aim of this paper is to give the general picture for any nonnegative vector α. Since the notations can become quite heavy, for the sake of simplicity, we start with the case $d = p = q = 1$. Several generalizations, including also the degenerate case in which α has some zero entries is treated in Section 6 via the observations in Subsection 2.1.3, which imply that the general analysis can be reduced to the case where all the entries of α are positive, that is $\alpha_j > 0$, $j = 1, \ldots, d$.

In the following, we denote by (n, α) the greater common divisor of n and α. i.e., $(n, \alpha) = \gcd(n, \alpha)$, by $n_\alpha = \frac{n}{(n,\alpha)}$, by $\check{\alpha} = \frac{\alpha}{(n,\alpha)}$, and by I_t the identity matrix of order t.

If we denote by C_n the classical circulant matrix (i.e. with $\alpha = 1$) and by $C_{n,\alpha}$ the α-circulant matrix generated by its elements, for generic n and α one verifies immediately that

$$C_{n,\alpha} = C_n Z_{n,\alpha}, \tag{10}$$

where

$$Z_{n,\alpha} = [\delta_{r-\alpha s}]_{r,s=0}^{n-1}, \qquad \delta_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Lemma 3.1. *Let n be any integer greater than 2 then*

$$Z_{n,\alpha} = \underbrace{\left[\widetilde{Z}_{n,\alpha}|\widetilde{Z}_{n,\alpha}|\cdots|\widetilde{Z}_{n,\alpha}\right]}_{(n,\alpha)\ times}, \tag{12}$$

where $Z_{n,\alpha}$ is the matrix defined in (11) and $\widetilde{Z}_{n,\alpha} \in \mathbb{C}^{n\times n_\alpha}$ is the submatrix of $Z_{n,\alpha}$ obtained by considering only its first n_α columns, that is

$$\widetilde{Z}_{n,\alpha} = Z_{n,\alpha}\begin{bmatrix} I_{n_\alpha} \\ 0 \end{bmatrix}. \tag{13}$$

Proof. Setting $\widetilde{Z}^{(0)}_{n,\alpha} = \widetilde{Z}_{n,\alpha}$ and denoting by $\widetilde{Z}^{(j)}_{n,\alpha} \in \mathbb{C}^{n\times n_\alpha}$ the $(j+1)$-th block-column of the matrix $Z_{n,\alpha}$ for $j = 0,\ldots,(n,\alpha)-1$, we find

$$Z_{n,\alpha} = \left[\underbrace{\widetilde{Z}^{(0)}_{n,\alpha}}_{n\times n_\alpha} \,|\, \underbrace{\widetilde{Z}^{(1)}_{n,\alpha}}_{n\times n_\alpha} \,|\cdots|\, \underbrace{\widetilde{Z}^{((n,\alpha)-1)}_{n,\alpha}}_{n\times n_\alpha}\right].$$

For $r = 0,1,\ldots,n-1$ and $s = 0,1,\ldots,n_\alpha-1$, we observe that

$$(\widetilde{Z}^{(j)}_{n,\alpha})_{r,s} = (Z_{n,\alpha})_{r,jn_\alpha+s},$$

and

$$\begin{aligned}(Z_{n,\alpha})_{r,jn_\alpha+s} &= \delta_{r-\alpha(jn_\alpha+s)} \\ &= \delta_{r-j\alpha n_\alpha-\alpha s} \\ &\underset{(a)}{=} \delta_{r-\alpha s} \\ &= (\widetilde{Z}^{(0)}_{n,\alpha})_{r,s} = (\widetilde{Z}_{n,\alpha})_{r,s},\end{aligned}$$

where $n_\alpha = \frac{n}{(n,\alpha)}$ and (a) is a consequence of the fact that $\frac{\alpha}{(n,\alpha)}$ is an integer greater than zero and so $j\alpha n_\alpha = j\frac{\alpha}{(n,\alpha)}n \equiv 0 \pmod{n}$. Thus we conclude that $\widetilde{Z}^{(j)}_{n,\alpha} = \widetilde{Z}^{(0)}_{n,\alpha} = \widetilde{Z}_{n,\alpha}$ for $j = 0,\ldots,(n,\alpha)-1$. □

Another useful fact is represented by the following equation

$$\widetilde{Z}_{n,\alpha} = \widetilde{Z}_{n,(n,\alpha)} Z_{n_\alpha,\breve{\alpha}}, \tag{14}$$

where $Z_{n_\alpha,\breve{\alpha}}$ is the matrix defined in (11) of dimension $n_\alpha \times n_\alpha$. Therefore

$$Z_{n_\alpha,\breve{\alpha}} = \left[\widehat{\delta}_{r-\breve{\alpha}s}\right]_{r,s=0}^{n_\alpha-1}, \qquad \widehat{\delta}_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{n_\alpha}, \\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

Relation (14) will be used later.

Proof. (of relation (14).) For $r = 0,1,\ldots,n-1$ and $s = 0,1,\ldots,n_\alpha-1$, we find

$$\begin{aligned}(\widetilde{Z}_{n,\alpha})_{r,s} &= \delta_{r-\alpha s} \\ &= \delta_{(r-\alpha s) \bmod n},\end{aligned}$$

and

$$
\begin{aligned}
(\widetilde{Z}_{n,(n,\alpha)} Z_{n_\alpha,\check{\alpha}})_{r,s} &= \sum_{l=0}^{n_\alpha-1} (\widetilde{Z}_{n,(n,\alpha)})_{r,l} (Z_{n_\alpha,\check{\alpha}})_{l,s} \\
&= \sum_{l=0}^{n_\alpha-1} \delta_{r-(n,\alpha)l} \widehat{\delta}_{l-\check{\alpha}s} \\
&\underset{(a)}{=} \delta_{r-(n,\alpha)\cdot(\check{\alpha}s) \bmod n_\alpha} \\
&= \delta_{r-(n,\alpha)\cdot\left(\frac{\alpha}{(n,\alpha)}s\right) \bmod n_\alpha} \\
&\underset{(b)}{=} \delta_{r-(\alpha s) \bmod n} \\
&= \delta_{(r-(\alpha s) \bmod n) \bmod n} \\
&= \delta_{(r-\alpha s) \bmod n},
\end{aligned}
$$

where

(a) holds true since there exists a unique $l \in \{0, 1, \ldots, n_\alpha - 1\}$ such that $l - \check{\alpha}s \equiv 0 \pmod{n_\alpha}$, that is, $l \equiv \check{\alpha}s \pmod{n_\alpha}$ and hence $\delta_{r-(n,\alpha)l} = \delta_{r-(n,\alpha)\cdot(\check{\alpha}s) \bmod n_\alpha}$;

(b) is due to the following property: if we have three integer numbers ρ, θ, and γ, then

$$\rho(\theta \bmod \gamma) = (\rho\theta) \bmod \rho\gamma.$$

□

Lemma 3.2. *If $\alpha \geq n$ then $Z_{n,\alpha} = Z_{n,\alpha^\circ}$ where α° is the unique integer which satisfies $\alpha = tn + \alpha^\circ$ with $0 \leq \alpha^\circ < n$ and $t \in \mathbb{N}$; $Z_{n,\alpha}$ is defined in (11).*

Remark 3.1. *One can define α° by: $\alpha^\circ := \alpha \bmod n$.*

Proof. From (11) we know that

$$Z_{n,\alpha} = [\delta_{r-\alpha c}]_{r,c=0}^{n-1}, \qquad \delta_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

For $r, s = 0, 1, \ldots, n-1$, one has

$$(Z_{n,\alpha})_{r,s} = \delta_{r-\alpha s} = \delta_{r-(tn+\alpha^\circ)s} = \delta_{r-\alpha^\circ s} = (Z_{n,\alpha^\circ})_{r,s},$$

since $tns \equiv 0 \pmod{n}$. Whence $Z_{n,\alpha} = Z_{n,\alpha^\circ}$. □

The previous lemma tells us that, for α-circulant matrices, we can consider only the case where $0 \leq \alpha < n$. In fact, if $\alpha \geq n$, from (10) we infer that

$$C_{n,\alpha} = C_n Z_{n,\alpha} = C_n Z_{n,\alpha^\circ} = C_{n,\alpha^\circ}.$$

Finally, it is worth noticing that the use of (5) and (10) implies that

$$C_{n,\alpha} = F_n D_n F_n^* Z_{n,\alpha}. \tag{16}$$

Formula (16) plays an important role for studying the singular values of the α-circulant matrices.

3.1 A characterization of $Z_{n,\alpha}$ in terms of Fourier matrices

Lemma 3.3. *Let F_n be the Fourier matrix of order n defined in (6) and let $\widetilde{Z}_{n,\alpha} \in \mathbb{C}^{n\times n_\alpha}$ be the matrix represented in (13). Then*

$$F_n\widetilde{Z}_{n,\alpha} = \frac{1}{\sqrt{(n,\alpha)}} I_{n,\alpha} F_{n_\alpha} Z_{n_\alpha,\breve{\alpha}}, \tag{17}$$

where $I_{n,\alpha} \in \mathbb{C}^{n\times n_\alpha}$ and

$$I_{n,\alpha} = \left.\begin{bmatrix} I_{n_\alpha} \\ \hline I_{n_\alpha} \\ \hline \vdots \\ \hline I_{n_\alpha} \end{bmatrix}\right\} (n,\alpha) \text{ times},$$

with I_{n_α} being the identity matrix of size n_α and $Z_{n_\alpha,\breve{\alpha}}$ as in (15).

Remark 3.2. $n = n_\alpha \cdot (n,\alpha)$.

Proof. (of Lemma 3.3.) Rewrite the Fourier matrix as

$$F_n = \frac{1}{\sqrt{n}} \left[\ f_0 \ \middle|\ f_1 \ \middle|\ f_2 \ \middle|\ \cdots \ \middle|\ f_{n-1}\ \right],$$

where f_k, $k = 0,1,2,\ldots,n-1$, is the $k-th$ column of the Fourier matrix of order n:

$$f_k = \left[e^{-\frac{2\pi i k j}{n}}\right]_{j=0}^{n-1} = \begin{bmatrix} e^{-\frac{2\pi i k\cdot 0}{n}} \\ e^{-\frac{2\pi i k\cdot 1}{n}} \\ e^{-\frac{2\pi i k\cdot 2}{n}} \\ \vdots \\ e^{-\frac{2\pi i k\cdot (n-1)}{n}} \end{bmatrix}. \tag{18}$$

From (14), we find

$$F_n\widetilde{Z}_{n,\alpha} = F_n\widetilde{Z}_{n,(n,\alpha)} Z_{n_\alpha,\breve{\alpha}} = \frac{1}{\sqrt{n}} \left[\ f_0 \ \middle|\ f_{1\cdot(n,\alpha)} \ \middle|\ f_{2\cdot(n,\alpha)} \ \middle|\ \cdots \ \middle|\ f_{(n_\alpha-1)\cdot(n,\alpha)}\ \right] Z_{n_\alpha,\breve{\alpha}} \in \mathbb{C}^{n\times n_\alpha}. \tag{19}$$

Indeed, for $k = 0,1,\ldots,n_\alpha-1$, $j = 0,1,\ldots,n-1$, one has

$$\left(F_n\widetilde{Z}_{n,(n,\alpha)}\right)_{j,k} = \sum_{l=0}^{n-1} (F_n)_{j,l} (\widetilde{Z}_{n,(n,\alpha)})_{l,k} = \sum_{l=0}^{n-1} \delta_{l-(n,\alpha)k} e^{-\frac{2\pi i j l}{n}}, \tag{20}$$

and, since $0 \le (n,\alpha)k \le n-(n,\alpha)$, there exists a unique $l_k \in \{0,1,2,\ldots,n-1\}$ such that $l_k - (n,\alpha)k \equiv 0 \pmod n$, so $l_k = (n,\alpha)k$. Consequently relation (20) implies

$$\left(F_n\widetilde{Z}_{n,(n,\alpha)}\right)_{j,k} = \delta_{l_k-(n,\alpha)k} e^{-\frac{2\pi i j l_k}{n}} = e^{-\frac{2\pi i j (n,\alpha)k}{n}} = \left(f_{(n,\alpha)k}\right)_j,$$

for all $0 \le j \le n-1$ and $0 \le k \le n_\alpha - 1$, and hence

$$F_n\widetilde{Z}_{n,(n,\alpha)} = \frac{1}{\sqrt{n}} \left[\ f_0 \ \middle|\ f_{1\cdot(n,\alpha)} \ \middle|\ f_{2\cdot(n,\alpha)} \ \middle|\ \cdots \ \middle|\ f_{(n_\alpha-1)\cdot(n,\alpha)}\ \right].$$

For $k = 0, 1, 2, \ldots, n_\alpha - 1$, we deduce

$$f_{(n,\alpha)k} = \left[e^{-\frac{2\pi i j (n,\alpha) k}{n}}\right]_{j=0}^{n-1} = \left[e^{-\frac{2\pi i j k}{n_\alpha}}\right]_{j=0}^{n-1},$$

and then, taking into account the equalities $n = (n,\alpha)\frac{n}{(n,\alpha)} = (n,\alpha)n_\alpha$, we can write

$$f_{(n,\alpha)k} = \begin{bmatrix} \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1} \\ \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=n_\alpha}^{2n_\alpha-1} \\ \vdots \\ \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=((n,\alpha)-1)n_\alpha}^{(n,\alpha)n_\alpha-1} \end{bmatrix}, \tag{21}$$

where

$$\left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1} = \begin{bmatrix} e^{-\frac{2\pi i k\cdot 0}{n_\alpha}} \\ e^{-\frac{2\pi i k\cdot 1}{n_\alpha}} \\ e^{-\frac{2\pi i k\cdot 2}{n_\alpha}} \\ \vdots \\ e^{-\frac{2\pi i k\cdot (n_\alpha-1)}{n_\alpha}} \end{bmatrix}. \tag{22}$$

According to formula (18), one observes that the vector in (22) is the $k-th$ column of the Fourier matrix F_{n_α}. Furthermore, for $l = 0, 1, 2, \ldots, (n,\alpha) - 1$, we find

$$\left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=l n_\alpha}^{(l+1)n_\alpha-1} = \begin{bmatrix} e^{-\frac{2\pi i k l n_\alpha}{n_\alpha}} \\ e^{-\frac{2\pi i k (l n_\alpha+1)}{n_\alpha}} \\ e^{-\frac{2\pi i k (l n_\alpha+2)}{n_\alpha}} \\ \vdots \\ e^{-\frac{2\pi i k (l n_\alpha+n_\alpha-1)}{n_\alpha}} \end{bmatrix} = e^{-2\pi i k l} \begin{bmatrix} e^{-\frac{2\pi i k\cdot 0}{n_\alpha}} \\ e^{-\frac{2\pi i k\cdot 1}{n_\alpha}} \\ e^{-\frac{2\pi i k\cdot 2}{n_\alpha}} \\ \vdots \\ e^{-\frac{2\pi i k\cdot (n_\alpha-1)}{n_\alpha}} \end{bmatrix} = \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1}. \tag{23}$$

Using (23), the expression of the vector in (21) becomes

$$f_{(n,\alpha)k} = \left.\begin{bmatrix} \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1} \\ \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1} \\ \vdots \\ \left[e^{-\frac{2\pi i k j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1} \end{bmatrix}\right\} (n,\alpha) \text{ times.} \tag{24}$$

Setting $\widetilde{f}_r = \left[e^{-\frac{2\pi i r j}{n_\alpha}}\right]_{j=0}^{n_\alpha-1}$, for $0 \le r \le n_\alpha - 1$, the Fourier matrix F_{n_α} of size n_α takes the form

$$F_{n_\alpha} = \frac{1}{\sqrt{n_\alpha}} \left[\; \widetilde{f}_0 \;\middle|\; \widetilde{f}_1 \;\middle|\; \widetilde{f}_2 \;\middle|\; \cdots \;\middle|\; \widetilde{f}_{n_\alpha-1} \;\right]. \tag{25}$$

From formula (22), the relation (24) can be expressed as

$$f_{(n,\alpha)k} = \left.\begin{bmatrix} \widetilde{f}_k \\ \widetilde{f}_k \\ \vdots \\ \widetilde{f}_k \end{bmatrix}\right\} (n,\alpha) \text{ times}, \qquad k = 0, \ldots, n_\alpha - 1,$$

and, as a consequence, formula (19) can be rewritten as

$$\begin{aligned}
F_n \widetilde{Z}_{n,\alpha} = F_n \widetilde{Z}_{n,(n,\alpha)} Z_{n_\alpha,\check{\alpha}} &= \frac{1}{\sqrt{n}} \left[\begin{array}{c|c|c|c|c} \widetilde{f}_0 & \widetilde{f}_1 & \widetilde{f}_2 & \cdots & \widetilde{f}_{n_\alpha-1} \\ \widetilde{f}_0 & \widetilde{f}_1 & \widetilde{f}_2 & \cdots & \widetilde{f}_{n_\alpha-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{f}_0 & \widetilde{f}_1 & \widetilde{f}_2 & \cdots & \widetilde{f}_{n_\alpha-1} \end{array}\right] Z_{n_\alpha,\check{\alpha}} \\
&= \frac{1}{\sqrt{(n,\alpha)n_\alpha}} \left[\begin{array}{c} \sqrt{n_\alpha} F_{n_\alpha} \\ \hline \sqrt{n_\alpha} F_{n_\alpha} \\ \hline \vdots \\ \hline \sqrt{n_\alpha} F_{n_\alpha} \end{array}\right] Z_{n_\alpha,\check{\alpha}} \\
&= \frac{1}{\sqrt{(n,\alpha)}} \left[\begin{array}{c} F_{n_\alpha} \\ \hline F_{n_\alpha} \\ \hline \vdots \\ \hline F_{n_\alpha} \end{array}\right] Z_{n_\alpha,\check{\alpha}} \\
&= \frac{1}{\sqrt{(n,\alpha)}} \left[\begin{array}{c} I_{n_\alpha} \\ \hline I_{n_\alpha} \\ \hline \vdots \\ \hline I_{n_\alpha} \end{array}\right] F_{n_\alpha} Z_{n_\alpha,\check{\alpha}} \\
&= \frac{1}{\sqrt{(n,\alpha)}} I_{n,\alpha} F_{n_\alpha} Z_{n_\alpha,\check{\alpha}}.
\end{aligned}$$

□

In the subsequent subsection, we will exploit Lemma 3.3 in order to characterize the singular values of the α-circulant matrices $C_{n,\alpha}$. Here we conclude the subsection with the following simple observations.

Remark 3.3. *In Lemma 3.3, if $(n,\alpha) = \alpha$, we have $n_\alpha = \frac{n}{(n,\alpha)} = \frac{n}{\alpha}$ and $\check{\alpha} = \frac{\alpha}{(n,\alpha)} = 1$; so the matrix $Z_{n_\alpha,\check{\alpha}} = Z_{n_\alpha,1}$, appearing in (17), is the identity matrix of dimension $\frac{n}{\alpha} \times \frac{n}{\alpha}$. The relation (17) becomes*

$$F_n \widetilde{Z}_{n,\alpha} = \frac{1}{\sqrt{\alpha}} I_{n,\alpha} F_{n_\alpha}.$$

The latter equation with $\alpha = 2$ and even n appear (and is crucial) in the multigrid literature; see [21], *equation (3.2), page 59 and, in slightly different form for the sine algebra of type I, see* [8], *Section 2.1.*

Remark 3.4. *If $(n,\alpha)=1$, Lemma 3.3 is trivial, because $n_\alpha=\frac{n}{(n,\alpha)}=n$, $\check{\alpha}=\frac{\alpha}{(n,\alpha)}=\alpha$, and so $\widetilde{Z}_{n,\alpha}=Z_{n,\alpha}$. The relation (17) becomes*

$$\begin{aligned} F_n\widetilde{Z}_{n,\alpha}=F_nZ_{n,\alpha} &= I_{n,\alpha}F_{n_\alpha}Z_{n_\alpha,\check{\alpha}} \\ &= F_nZ_{n,\alpha}, \end{aligned}$$

since the matrix $I_{n,\alpha}$ reduces by its definition to the identity matrix of order n.

Remark 3.5. *Lemma 3.3 is true also if, instead of F_n and F_{n_α}, we put F_n^* and $F_{n_\alpha}^*$, respectively, because $F_n^*=\overline{F_n}$. In fact there is no transposition, but only conjugation.*

3.2 Characterization of the singular values of the α-circulant matrices

Now we link the singular values of α-circulant matrices with the eigenvalues of its circulant counterpart C_n. This is nontrivial given the multiplicative relation $C_{n,\alpha}=C_nZ_{n,\alpha}$.

Having in mind the definition of the diagonal matrix D_n given in (6), we start by setting

$$D_n^*D_n=\mathrm{diag}(|D_n|^2_{s,s};\ s=0,1,\ldots,n-1)=\mathrm{diag}(d_s;\ s=0,1,\ldots,n-1)=\bigoplus_{l=1}^{(n,\alpha)}\Delta_l,$$

$$J_{(n,\alpha)}\otimes I_{n_\alpha}=\underbrace{[I_{n,\alpha}|I_{n,\alpha}|\cdots|I_{n,\alpha}]}_{(n,\alpha)\ \text{times}}=\left.\left[\begin{array}{c|c|c|c} I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \end{array}\right]\right\}(n,\alpha)\ \text{times}, \tag{26}$$

where

$$d_s=|D_n|^2_{s,s}=(D_n)_{s,s}\cdot\overline{(D_n)_{s,s}},\quad D_n\ \text{defined in (6)},\ s=0,1,\ldots,n-1, \tag{27}$$

$$\Delta_l=\begin{bmatrix} d_{(l-1)n_\alpha} & & & \\ & d_{(l-1)n_\alpha+1} & & \\ & & \ddots & \\ & & & d_{(l-1)n_\alpha+n_\alpha-1}\end{bmatrix}\in\mathbb{C}^{n_\alpha\times n_\alpha};\ l=1,2,\ldots,(n,\alpha),$$

$$J_{(n,\alpha)}=\left.\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1\end{bmatrix}\right\}(n,\alpha)\ \text{times}. \tag{28}$$

We now exploit relation (12) and Lemma 3.3, and we obtain that

$$\begin{aligned} F_nZ_{n,\alpha} &= F_n\left[\widetilde{Z}_{n,\alpha}|\widetilde{Z}_{n,\alpha}|\cdots|\widetilde{Z}_{n,\alpha}\right] \\ &= \left[F_n\widetilde{Z}_{n,\alpha}|F_n\widetilde{Z}_{n,\alpha}|\cdots|F_n\widetilde{Z}_{n,\alpha}\right] \\ &= \frac{1}{\sqrt{(n,\alpha)}}\left[I_{n,\alpha}F_{n_\alpha}Z_{n_\alpha,\check{\alpha}}|I_{n,\alpha}F_{n_\alpha}Z_{n_\alpha,\check{\alpha}}|\cdots|I_{n,\alpha}F_{n_\alpha}Z_{n_\alpha,\check{\alpha}}\right] \\ &= \frac{1}{\sqrt{(n,\alpha)}}[I_{n,\alpha}|I_{n,\alpha}|\cdots|I_{n,\alpha}]\left.\begin{bmatrix} F_{n_\alpha}Z_{n_\alpha,\check{\alpha}} & & & \\ & F_{n_\alpha}Z_{n_\alpha,\check{\alpha}} & & \\ & & \ddots & \\ & & & F_{n_\alpha}Z_{n_\alpha,\check{\alpha}}\end{bmatrix}\right\}(n,\alpha)\ \text{times} \\ &= \frac{1}{\sqrt{(n,\alpha)}}[I_{n,\alpha}|I_{n,\alpha}|\cdots|I_{n,\alpha}]\left(I_{(n,\alpha)}\otimes F_{n_\alpha}Z_{n_\alpha,\check{\alpha}}\right), \end{aligned} \tag{29}$$

where $I_{(n,\alpha)}$ is the identity matrix of order (n,α). Furthermore,

$$\begin{aligned}
C_{n,\alpha}^* C_{n,\alpha} &= (F_n D_n F_n^* Z_{n,\alpha})^* (F_n D_n F_n^* Z_{n,\alpha}) \\
&= Z_{n,\alpha}^* F_n D_n^* F_n^* F_n D_n F_n^* Z_{n,\alpha} \\
&= Z_{n,\alpha}^* F_n D_n^* D_n F_n^* Z_{n,\alpha} \\
&= (F_n^* Z_{n,\alpha})^* D_n^* D_n F_n^* Z_{n,\alpha}. \qquad (30)
\end{aligned}$$

From (29) and (26), we plainly infer the following relations

$$\begin{aligned}
(F_n^* Z_{n,\alpha})^* &= \left(\frac{1}{\sqrt{(n,\alpha)}} [I_{n,\alpha}|I_{n,\alpha}|\cdots|I_{n,\alpha}] \left(I_{(n,\alpha)} \otimes F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}} \right) \right)^* \\
&= \frac{1}{\sqrt{(n,\alpha)}} \left(I_{(n,\alpha)} \otimes F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}} \right)^* \left(J_{(n,\alpha)} \otimes I_{n_\alpha} \right) \\
&= \frac{1}{\sqrt{(n,\alpha)}} \left(I_{(n,\alpha)} \otimes Z_{n_\alpha,\breve{\alpha}}^* F_{n_\alpha} \right) \left(J_{(n,\alpha)} \otimes I_{n_\alpha} \right), \\
F_n^* Z_{n,\alpha} &= \frac{1}{\sqrt{(n,\alpha)}} [I_{n,\alpha}|I_{n,\alpha}|\cdots|I_{n,\alpha}] \left(I_{(n,\alpha)} \otimes F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}} \right) \\
&= \frac{1}{\sqrt{(n,\alpha)}} \left(J_{(n,\alpha)} \otimes I_{n_\alpha} \right) \left(I_{(n,\alpha)} \otimes F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}} \right).
\end{aligned}$$

Hence

$$C_{n,\alpha}^* C_{n,\alpha} = \left(I_{(n,\alpha)} \otimes Z_{n_\alpha,\breve{\alpha}}^* F_{n_\alpha} \right) \left(J_{(n,\alpha)} \otimes I_{n_\alpha} \right) \frac{1}{(n,\alpha)} D_n^* D_n \left(J_{(n,\alpha)} \otimes I_{n_\alpha} \right) \left(I_{(n,\alpha)} \otimes F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}} \right).$$

Now using the properties of the tensorial product

$$\begin{aligned}
&(I_{(n,\alpha)} \otimes Z_{n_\alpha,\breve{\alpha}}^* F_{n_\alpha})(I_{(n,\alpha)} \otimes F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}}) \\
&\qquad = I_{(n,\alpha)} I_{(n,\alpha)} \otimes Z_{n_\alpha,\breve{\alpha}}^* F_{n_\alpha} F_{n_\alpha}^* Z_{n_\alpha,\breve{\alpha}} \\
&\qquad = I_{(n,\alpha)} I_{(n,\alpha)} \otimes Z_{n_\alpha,\breve{\alpha}}^* Z_{n_\alpha,\breve{\alpha}} \\
&\qquad = I_{(n,\alpha)} I_{(n,\alpha)} \otimes I_{n_\alpha} = I_n,
\end{aligned}$$

and from a similarity argument, one deduces that the eigenvalues of $C_{n,\alpha}^* C_{n,\alpha}$ are the eigenvalues

of the matrix

$$
\begin{aligned}
&\left(J_{(n,\alpha)} \otimes I_{n_\alpha}\right) \frac{1}{(n,\alpha)} D_n^* D_n \left(J_{(n,\alpha)} \otimes I_{n_\alpha}\right) \\
&= \frac{1}{(n,\alpha)} \left[\begin{array}{c|c|c|c} I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \end{array}\right] \begin{bmatrix} \Delta_1 & & & \\ & \Delta_2 & & \\ & & \ddots & \\ & & & \Delta_{(n,\alpha)} \end{bmatrix} \left[\begin{array}{c|c|c|c} I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \end{array}\right] \\
&= \frac{1}{(n,\alpha)} \left[\begin{array}{c|c|c|c} I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline I_{n_\alpha} & I_{n_\alpha} & \cdots & I_{n_\alpha} \end{array}\right] \left[\begin{array}{c|c|c|c} \Delta_1 & \Delta_1 & \cdots & \Delta_1 \\ \hline \Delta_2 & \Delta_2 & \cdots & \Delta_2 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \Delta_{(n,\alpha)} & \Delta_{(n,\alpha)} & \cdots & \Delta_{(n,\alpha)} \end{array}\right] \\
&= \frac{1}{(n,\alpha)} \left[\begin{array}{c|c|c|c} \sum_{l=1}^{(n,\alpha)} \Delta_l & \sum_{l=1}^{(n,\alpha)} \Delta_l & \cdots & \sum_{l=1}^{(n,\alpha)} \Delta_l \\ \hline \sum_{l=1}^{(n,\alpha)} \Delta_l & \sum_{l=1}^{(n,\alpha)} \Delta_l & \cdots & \sum_{l=1}^{(n,\alpha)} \Delta_l \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \sum_{l=1}^{(n,\alpha)} \Delta_l & \sum_{l=1}^{(n,\alpha)} \Delta_l & \cdots & \sum_{l=1}^{(n,\alpha)} \Delta_l \end{array}\right] \\
&= \frac{1}{(n,\alpha)} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{(n,\alpha) \text{ times}} \otimes \left(\sum_{l=1}^{(n,\alpha)} \Delta_l \right).
\end{aligned}
$$

Therefore, from (28), we infer that

$$
\mathrm{Eig}(C_{n,\alpha}^* C_{n,\alpha}) = \frac{1}{(n,\alpha)} \mathrm{Eig} \left(J_{(n,\alpha)} \otimes \sum_{l=1}^{(n,\alpha)} \Delta_l \right), \tag{31}
$$

where

$$
\frac{1}{(n,\alpha)} \mathrm{Eig}(J_{(n,\alpha)}) = \{0, 1\}. \tag{32}
$$

Here we must observe that $\frac{1}{(n,\alpha)} J_{(n,\alpha)}$ is a matrix of rank 1, so it has all eigenvalues equal to zero except one eigenvalue equal to 1. In fact note that the trace of a matrix is, by definition, the sum of its eigenvalues: in our case the trace is $(n,\alpha) \cdot \frac{1}{(n,\alpha)} = 1$ and hence the only nonzero eigenvalue is necessarily equal to 1. Moreover

$$
\begin{aligned}
\sum_{l=1}^{(n,\alpha)} \Delta_l &= \sum_{l=1}^{(n,\alpha)} \mathrm{diag}(d_{(l-1)n_\alpha + j};\ j = 0, 1, \ldots, n_\alpha - 1) \\
&= \mathrm{diag} \left(\sum_{l=1}^{(n,\alpha)} d_{(l-1)n_\alpha + j};\ j = 0, 1, \ldots, n_\alpha - 1 \right).
\end{aligned}
$$

Consequently, since $\sum_{l=1}^{(n,\alpha)} \Delta_l$ is a diagonal matrix, we have

$$\mathrm{Eig}\left(\sum_{l=1}^{(n,\alpha)} \Delta_l\right) = \left\{\sum_{l=1}^{(n,\alpha)} d_{(l-1)n_\alpha+j};\ j=0,1,\ldots,n_\alpha-1\right\}, \tag{33}$$

where d_k are defined in (27).

Finally, by exploiting basic properties of the tensor product, we know that the eigenvalues of a tensor product of two square matrices $A\otimes B$ are given by all possible products of eigenvalues of A of order p and of eigenvalues of B of order q, that is $\lambda(A\otimes B)=\lambda_j(A)\lambda_k(B)$ for $j=1,\ldots,p$ and $k=1,\ldots,q$. Therefore, by taking into consideration (31), (32), and (33), we find

$$\lambda_j(C_{n,\alpha}^* C_{n,\alpha}) = \sum_{l=1}^{(n,\alpha)} d_{(l-1)n_\alpha+j}, \quad j=0,1,\ldots,n_\alpha-1, \tag{34}$$

$$\lambda_j(C_{n,\alpha}^* C_{n,\alpha}) = 0, \quad j=n_\alpha,\ldots,n-1. \tag{35}$$

From (34), (35) and (1), one obtains that the singular values of an α-circulant matrix $C_{n,\alpha}$ are given by

$$\sigma_j(C_{n,\alpha}) = \sqrt{\sum_{l=1}^{(n,\alpha)} d_{(l-1)n_\alpha+j}}, \quad j=0,1,\ldots,n_\alpha-1, \tag{36}$$

$$\sigma_j(C_{n,\alpha}) = 0, \quad j=n_\alpha,\ldots,n-1,$$

where the values d_k, $k=0,\ldots,n-1$, are defined in (27).

3.3 Special cases and observations

In this subsection we consider some special cases and we furnish a further link between the eigenvalues of circulant matrices and the singular values of α-circulants. In the case where $(n,\alpha)=1$, we have $n_\alpha=\frac{n}{(n,\alpha)}=n$. Hence the formula (36) becomes

$$\sigma_j(C_{n,\alpha})=\sqrt{d_j}, \quad j=0,1,\ldots,n-1.$$

In other words the singular values of $C_{n,\alpha}$ coincide with those of C_n (this is expected since $Z_{n,\alpha}$ is a permutation matrix) and in particular with the moduli of the eigenvalues of C_n.

Concerning the eigenvalues of circulant matrices it should be observed that formula (6) can be interpreted in function terms as the evaluation of a polynomial at the grid points given by the n-th roots of the unity. This is a standard observation because the Fourier matrix is a special instance of the classical Vandermonde matrices when the knots are exactly all the n-th roots of the unity.

Therefore, defining the polynomial $p(t)=\sum_{k=0}^{n-1} a_k e^{ikt}$, it is trivial to observe that the eigenvalues of $C_n=F_nD_nF_n^*$ are given by

$$\lambda_j(C_n)=p\left(\frac{2\pi j}{n}\right), \quad j=0,\ldots,n-1.$$

The question that naturally arises is how to connect the expression in (36) of the nontrivial singular values of $C_{n,\alpha}$ with the polynomial p. The answer is somehow intriguing and can be resumed in the following formula which could be of interest in the multigrid community (see Section 5)

$$\sigma_j(C_{n,\alpha}) = \sqrt{\sum_{l=0}^{(n,\alpha)-1} |p|^2\left(\frac{x_j+2\pi l}{(n,\alpha)}\right)}, \quad x_j=\frac{2\pi j}{n_\alpha}, \quad j=0,1,\ldots,n_\alpha-1. \tag{37}$$

In addition if α is fixed and a sequence of integers n is chosen so that $(n,\alpha) > 1$ for n large enough, then $\{C_{n,\alpha}\} \sim_\sigma (0, G)$ for a proper set G. If the sequence of n is chosen so that n and α are coprime for all n large enough, then the existence of the distribution is related to the smoothness properties of a function f such that $\{a_k\}$ can be interpreted as the sequence of its Fourier coefficients (see e.g. [20]). From the above reasoning it is clear that, if n is allowed to be vary among all the positive integer numbers, then $\{C_{n,\alpha}\}$ does not possess a joint singular value distribution.

4 Singular values of α-Toeplitz matrices

For $p = q = d = 1$, we recall that the α-Toeplitz matrices of dimension $n \times n$ are defined as

$$T_{n,\alpha} = [a_{r-\alpha c}]_{r,c=0}^{n-1}, \tag{38}$$

where the quantities $r - \alpha s$ are not reduced modulus n. In analogy with the case of $\alpha = 1$, the elements a_j are the Fourier coefficients of some function f in $L^1(Q)$, with $Q = (-\pi, \pi)$, i.e., $a_j = \tilde{f}_j$ as in (4) with $d = 1$. If we denote by T_n the classical Toeplitz matrix generated by the function $f \in L^1(Q)$, $T_n = [a_{r-c}]_{r,c=0}^{n-1}$, $a_j = \tilde{f}_j$ defined as in (4), and by $T_{n,\alpha}$ the α-Toeplitz matrix generated by the same function, one verifies immediately for n and α generic that

$$T_{n,\alpha} = \left[\widehat{T}_{n,\alpha} | \mathcal{T}_{n,\alpha}\right] = \left[T_n \widehat{Z}_{n,\alpha} | \mathcal{T}_{n,\alpha}\right], \tag{39}$$

where $\widehat{T}_{n,\alpha} \in \mathbb{C}^{n\times\mu_\alpha}$, $\mu_\alpha = \left\lceil \frac{n}{\alpha} \right\rceil$, is the matrix $T_{n,\alpha}$ defined in (38) by considering only the μ_α first columns, $\mathcal{T}_{n,\alpha} \in \mathbb{C}^{n\times(n-\mu_\alpha)}$ is the matrix $T_{n,\alpha}$ defined in (38) by considering only the $n - \mu_\alpha$ last columns, and $\widehat{Z}_{n,\alpha}$ is the matrix defined in (11) by considering only the μ_α first columns.

Proof. (of relation (39).) For $r = 0, 1, \ldots, n-1$ and $s = 0, 1, \ldots, \mu_\alpha - 1$, one has

$$\begin{aligned}
(\widehat{T}_{n,\alpha})_{r,s} &= (T_n)_{r,\alpha s}, \\
(\widehat{Z}_{n,\alpha})_{r,s} &= \delta_{r-\alpha s},
\end{aligned}$$

and

$$\begin{aligned}
(T_n \widehat{Z}_{n,\alpha})_{r,s} &= \sum_{l=0}^{n-1} (T_n)_{r,l} (\widehat{Z}_{n,\alpha})_{l,s} \\
&= \sum_{l=0}^{n-1} \delta_{l-\alpha s} (T_n)_{r,l} \\
&\underset{\text{(a)}}{=} (T_n)_{r,\alpha s} \\
&= (\widehat{T}_{n,\alpha})_{r,s},
\end{aligned}$$

where (a) follows because there exists a unique $l \in \{0, 1, \ldots, n-1\}$ such that $l - \alpha s \equiv 0 \pmod{n}$, that is, $l \equiv \alpha s \pmod{n}$, and, since $0 \le \alpha s \le n-1$, we obtain $l = \alpha s$. □

If we take the matrix $\widehat{T}_{n,\alpha}$ of size $n \times (\mu_\alpha + 1)$, then relation (39) is no longer true. In reality, looking at the $(\mu_\alpha + 1)$-th column of the α-Toeplitz we observe Fourier coefficients with indices which are not present (less or equal to $-n$) in the Toeplitz matrix T_n. More precisely,

$$(T_{n,\alpha})_{0,\mu_\alpha} = a_{0-\alpha\mu_\alpha} = a_{-\alpha\mu_\alpha}, \qquad \text{and } -\alpha\mu_\alpha \le -n.$$

It follows that μ_α is the maximum number of columns for which relation (39) is true.

4.1 Some preparatory results

We begin with some preliminary notations and definitions.

Definition 4.1. *Suppose a sequence of matrices $\{A_n\}_n$ of size d_n is given. We say that $\{\{B_{n,m}\}_n : m \geq 0\}$, $B_{n,m}$ of size d_n, $m \in \mathbb{N}$, is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ if, for all sufficiently large $m \in \mathbb{N}$, the following splitting holds:*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m} \quad \text{for all } n > n_m, \tag{40}$$

with

$$\mathrm{Rank}(R_{n,m}) \leq d_n\, c(m), \quad \|N_{n,m}\| \leq \omega(m), \tag{41}$$

where $\|\cdot\|$ is the spectral norm (largest singular value), n_m, $c(m)$ and $\omega(m)$ depend only on m and, moreover,

$$\lim_{m\to\infty} \omega(m) = 0, \quad \lim_{m\to\infty} c(m) = 0. \tag{42}$$

Proposition 4.1. [14] *Let $\{d_n\}_n$ be an increasing sequence of natural numbers. Suppose a sequence formed by matrices $\{A_n\}_n$ of size d_n is given such that $\{\{B_{n,m}\}_n :\ m \geq 0\}$, $m \in \hat{\mathbb{N}} \subset \mathbb{N}$, $\#\hat{\mathbb{N}} = \infty$, is an a.c.s. for $\{A_n\}_n$ in the sense of Definition 4.1. Suppose that $\{B_{n,m}\}_n \sim_\sigma (\theta_m, G)$ and that θ_m converges in measure to the measurable function θ over G. Then necessarily*

$$\{A_n\}_n \sim_\sigma (\theta, G), \tag{43}$$

(see Definition 2.1).

Proposition 4.2. [14, 17] *If $\{A_n\}_n$ and $\{B_n\}_n$ are two sequences of matrices of strictly increasing dimension, such that $\{A_n\}_n \sim_\sigma (\theta, G)$ and $\{B_n\}_n \sim_\sigma (0, G)$, then*

$$\{A_n + B_n\}_n \sim_\sigma (\theta, G).$$

Proposition 4.3. [14] *Let $f, g \in L^1(Q^d)$, $Q = (-\pi, \pi)$, and let $\{T_n(f)\}_n$ and $\{T_n(g)\}_n$ be the two sequences of Toeplitz matrices generated by f and g, respectively. The following distribution result is true*

$$\{T_n(f)T_n(g)\}_n \sim_\sigma (fg, Q^d).$$

Lemma 4.1. *Let f be a measurable complex-valued function on a set K, and consider the measurable function $\sqrt{|f|} : K \to \mathbb{R}^+$. Let $\{A_{n,m}\}$, with $A_{n,m} \in \mathbb{C}^{d_n \times d'_n}$, $d'_n \leq d_n$, be a sequence of matrices of strictly increasing dimension: $d'_n < d'_{n+1}$ and $d_n \leq d_{n+1}$. If the sequence of matrices $\{A^*_{n,m}A_{n,m}\}$, with $A^*_{n,m}A_{n,m} \in \mathbb{C}^{d'_n \times d'_n}$ and $d'_n < d'_{n+1}$, is distributed in the singular value sense as the function f over a proper set $G \subset K$ in the sense of Definition 2.1, then the sequence $\{A_{n,m}\}$ is distributed in the singular value sense as the function $\sqrt{|f|}$ over the same G.*

Proof. From the singular value decomposition (SVD), we can write $A_{n,m}$ as

$$A_{n,m} = U\Sigma V^* = U \left[\begin{array}{cccc} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{d'_n} \\ \hline & & 0 & \end{array}\right] V^*,$$

with U and V unitary matrices $U \in \mathbb{C}^{d_n \times d_n}$, $V \in \mathbb{C}^{d'_n \times d'_n}$ and $\Sigma \in \mathbb{R}^{d_n \times d'_n}$, $\sigma_j \geq 0$; by multiplying $A^*_{n,m}A_{n,m}$ we obtain:

$$\begin{aligned} A^*_{n,m}A_{n,m} = V\Sigma^T U^* U \Sigma V^* &= V\Sigma^T\Sigma V^* = V\Sigma^{(2)}V^* \\ &= V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_{d'_n}^2 \end{bmatrix} V^*, \end{aligned} \tag{44}$$

with V unitary matrix $V \in \mathbb{C}^{d'_n \times d'_n}$ and $\Sigma^{(2)} \in \mathbb{R}^{d'_n \times d'_n}$, $\sigma_j^2 \geq 0$; we observe that (44) is an SVD for $A^*_{n,m}A_{n,m}$, that is, the singular values $\sigma_j(A^*_{n,m}A_{n,m})$ of $A^*_{n,m}A_{n,m}$ are the square of singular values $\sigma_j(A_{n,m})$ of $A_{n,m}$. Since $\{A^*_{n,m}A_{n,m}\} \sim_\sigma (f, G)$, by definition it hold that for every $F \in \mathcal{C}_0(\mathbb{R}_0^+)$

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{d'_n} \sum_{i=1}^{d'_n} F\left(\sigma_i(A^*_{n,m}A_{n,m})\right) &= \frac{1}{\mu(G)} \int_G F\left(|f(t)|\right) dt \\ &= \frac{1}{\mu(G)} \int_G H\left(\sqrt{|f(t)|}\right) dt, \end{aligned} \tag{45}$$

where H is such that $F = H \circ \sqrt{\cdot}$; but, owing to $\sigma_j(A_{n,m}) = \sqrt{\sigma_j(A^*_{n,m}A_{n,m})}$ we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{d'_n} \sum_{i=1}^{d'_n} F\left(\sigma_i(A^*_{n,m}A_{n,m})\right) &= \lim_{n\to\infty} \frac{1}{d'_n} \sum_{i=1}^{d'_n} F\left(\sigma_i^2(A_{n,m})\right) \\ &= \lim_{n\to\infty} \frac{1}{d'_n} \sum_{i=1}^{d'_n} H\left(\sigma_i(A_{n,m})\right). \end{aligned} \tag{46}$$

From (45) and (46) we obtain

$$\lim_{n\to\infty} \frac{1}{d'_n} \sum_{i=1}^{d'_n} H\left(\sigma_i(A_{n,m})\right) = \frac{1}{\mu(G)} \int_G H\left(\sqrt{|f(t)|}\right) dt, \tag{47}$$

for every $H \in \mathcal{C}_0(\mathbb{R}_0^+)$, so $\{A_{n,m}\} \sim_\sigma (\sqrt{|f(t)|}, G)$. □

Lemma 4.2. *Let $\{A_n\}_n$ and $\{Q_n\}_n$ be two sequences of matrices of strictly increasing dimension ($A_n, Q_n \in \mathbb{C}^{d_n \times d_n}$, $d_n < d_{n+1}$), where Q_n are all unitary matrices ($Q_nQ_n^* = I$). If $\{A_n\}_n \sim_\sigma (0, G)$ then $\{A_nQ_n\}_n \sim_\sigma (0, G)$ and $\{Q_nA_n\}_n \sim_\sigma (0, G)$.*

Proof. Putting $B_n = A_nQ_n$, assuming that

$$A_n = U_n\Sigma_nV_n,$$

is an SVD for A_n, and taking into account that the product of two unitary matrices is still a unitary matrix, we deduce that the writing

$$B_n = A_nQ_n = U_n\Sigma_nV_nQ_n = U_n\Sigma_n\widehat{V}_n,$$

is an SVD for B_n. The latter implies that A_n and B_n have exactly the same singular values, so that the two sequences $\{A_n\}_n$ and $\{B_n\}_n$ are distributed in the same way. □

Lemma 4.3. *Let $\{A_n\}_n$ and $\{Q_n\}_n$ be two sequences of matrices of strictly increasing dimension ($A_n, Q_n \in \mathbb{C}^{d_n \times d_n}$, $d_n < d_{n+1}$). If $\{A_n\}_n \sim_\sigma (0, G)$ and $\|Q_n\| \leq M$ for some nonnegative constant M independent of n, then $\{A_n Q_n\}_n \sim_\sigma (0, G)$ and $\{Q_n A_n\}_n \sim_\sigma (0, G)$.*

Proof. Since $\{A_n\}_n \sim_\sigma (0, G)$, then $\{0_n\}_n$ (sequence of zero matrices) is an *a.c.s.* for $\{A_n\}_n$; this means (by Definition (4.1)) that we can write, for every m sufficiently large, $m \in \mathbb{N}$

$$A_n = 0_n + R_{n,m} + N_{n,m}, \qquad \forall n > n_m, \tag{48}$$

with

$$\mathrm{Rank}(R_{n,m}) \leq d_n c(m), \qquad \|N_{n,m}\| \leq \omega(m),$$

where $n_m \geq 0$, $c(m)$ and $\omega(m)$ depend only on m and, moreover

$$\lim_{m\to\infty} c(m) = 0, \qquad \lim_{m\to\infty} \omega(m) = 0.$$

Now consider the matrix $A_n Q_n$; from (48) we obtain

$$A_n Q_n = 0_n + R_{n,m} Q_n + N_{n,m} Q_n, \qquad \forall n > n_m,$$

with

$$\begin{aligned}
&\mathrm{Rank}(R_{n,m} Q_n) \leq \min\{\mathrm{Rank}(R_{n,m}), \mathrm{Rank}(Q_n)\} \leq \mathrm{Rank}(R_{n,m}) \leq d_n c(m),\\
&\|N_{n,m} Q_n\| \leq \|N_{n,m}\| \|Q_n\| \leq M\omega(m),
\end{aligned}$$

where

$$\lim_{m\to\infty} c(m) = 0, \qquad \lim_{m\to\infty} M\omega(m) = 0,$$

then $\{0_n\}_n$ is an *a.c.s.* for the sequence $\{A_n Q_n\}_n$ and, by Proposition 4.1, $\{A_n Q_n\}_n \sim_\sigma (0, G)$. □

4.2 Singular value distribution for the α-Toeplitz sequences

As stated in formula (39), the matrix $T_{n,\alpha}$ can be written as

$$\begin{aligned}
T_{n,\alpha} &= \left[T_n \widehat{Z}_{n,\alpha} | \mathcal{T}_{n,\alpha} \right]\\
&= \left[\; T_n \widehat{Z}_{n,\alpha} \; \middle| \; 0 \; \right] + \left[\; 0 \; \middle| \; \mathcal{T}_{n,\alpha} \; \right].
\end{aligned} \tag{49}$$

To find the distribution in the singular value sense of the sequence $\{T_{n,\alpha}\}_n$, the idea is to study separately the distribution of the two sequences $\{[T_n \widehat{Z}_{n,\alpha}|0]\}_n$ and $\{[0|\mathcal{T}_{n,\alpha}]\}_n$, to prove $\{[0|\mathcal{T}_{n,\alpha}]\}_n \sim (0, G)$, and then apply Proposition 4.2.

4.2.1 Singular value distribution for the sequence $\{[T_n \widehat{Z}_{n,\alpha}|0]\}_n$

Since $T_n \widehat{Z}_{n,\alpha} \in \mathbb{C}^{n \times \mu_\alpha}$ and $[T_n \widehat{Z}_{n,\alpha}|0] \in \mathbb{C}^{n \times n}$, the matrix $[T_n \widehat{Z}_{n,\alpha}|0]$ has $n - \mu_\alpha$ singular values equal to zero and the remaining μ_α equal to those of $T_n \widehat{Z}_{n,\alpha}$; to study the distribution in the singular value sense of this sequence of non-square matrices, we use Lemma 4.1: consider the α-Toeplitz matrix

"truncated" $\widehat{T}_{n,\alpha} = T_n(f)\widehat{Z}_{n,\alpha}$, where the elements of the Toeplitz matrix $T_n(f) = [a_{r-c}]_{r,c=0}^{n-1}$ are the Fourier coefficients of a function f in $L^1(Q)$, $Q = (-\pi, \pi)$, then we have

$$\begin{aligned}
\widehat{T}_{n,\alpha}^{*}\widehat{T}_{n,\alpha} &= (T_n(f)\widehat{Z}_{n,\alpha})^{*}T_n(f)\widehat{Z}_{n,\alpha} = \widehat{Z}_{n,\alpha}^{*}T_n(f)^{*}T_n(f)\widehat{Z}_{n,\alpha} \\
&= \widehat{Z}_{n,\alpha}^{*}T_n(\overline{f})T_n(f)\widehat{Z}_{n,\alpha}.
\end{aligned} \tag{50}$$

We provide in detail the analysis in the case where $f \in L^2(Q)$. The general setting in which $f \in L^1(Q)$ can be obtained by approximation and density arguments as done in [14]. From Proposition 4.3 if $f \in L^2(Q) \subset L^1(Q)$ (that is $|f|^2 \in L^1(Q)$), then $\{T_n(\overline{f})T_n(f)\}_n \sim_\sigma (|f|^2, Q)$. Consequently, for every m sufficiently large, $m \in \mathbb{N}$, the use of Proposition 4.1 implies

$$T_n(\overline{f})T_n(f) = T_n(|f|^2) + R_{n,m} + N_{n,m}, \qquad \forall n > n_m,$$

with

$$\mathrm{Rank}(R_{n,m}) \le nc(m), \qquad \|N_{n,m}\| \le \omega(m),$$

where $n_m \ge 0$, $c(m)$ and $\omega(m)$ depend only on m and, moreover

$$\lim_{m\to\infty} c(m) = 0, \qquad \lim_{m\to\infty} \omega(m) = 0.$$

Therefore (50) becomes

$$\begin{aligned}
\widehat{T}_{n,\alpha}^{*}\widehat{T}_{n,\alpha} &= \widehat{Z}_{n,\alpha}^{*}(T_n(|f|^2) + R_{n,m} + N_{n,m})\widehat{Z}_{n,\alpha} \\
&= \widehat{Z}_{n,\alpha}^{*}T_n(|f|^2)\widehat{Z}_{n,\alpha} + \widehat{Z}_{n,\alpha}^{*}R_{n,m}\widehat{Z}_{n,\alpha} + \widehat{Z}_{n,\alpha}^{*}N_{n,m}\widehat{Z}_{n,\alpha} \\
&= \widehat{Z}_{n,\alpha}^{*}T_n(|f|^2)\widehat{Z}_{n,\alpha} + \widehat{R}_{n,m,\alpha} + \widehat{N}_{n,m,\alpha},
\end{aligned} \tag{51}$$

with

$$\mathrm{Rank}(\widehat{R}_{n,m,\alpha}) \le \min\{\mathrm{Rank}(\breve{Z}_{n,\alpha}), \mathrm{Rank}(R_{n,m})\} \le \mathrm{Rank}(R_{n,m}) \le nc(m), \tag{52}$$

$$\|\widehat{N}_{n,m,\alpha}\| \le 2\|\breve{Z}_{n,\alpha}\|\|N_{n,m}\| \le 2\omega(m), \tag{53}$$

and

$$\lim_{m\to\infty} c(m) = 0, \qquad \lim_{m\to\infty} 2\omega(m) = 0,$$

where in (52) and (53), $\breve{Z}_{n,\alpha} = [\widehat{Z}_{n,\alpha}|0] \in \mathbb{C}^{n\times n}$. In other words $\breve{Z}_{n,\alpha}$ is the matrix $\widehat{Z}_{n,\alpha}$ supplemented by an appropriate number of zero columns in order to make it square. Furthermore, it is worth noticing that $\|\widehat{Z}_{n,\alpha}\| = \|\widehat{Z}_{n,\alpha}^{*}\| = 1$, because $\widehat{Z}_{n,\alpha}$ is a submatrix of the identity: we have used the latter relations in (53).

Now, consider the matrix $\widehat{Z}_{n,\alpha}^{*}T_n(|f|^2)\widehat{Z}_{n,\alpha} \in \mathbb{C}^{\mu_\alpha\times\mu_\alpha}$, with $\mu_\alpha = \lceil \frac{n}{\alpha} \rceil$, $f \in L^2(Q) \subset L^1(Q)$ (so $|f|^2 \in L^1(Q)$). From (39), setting $T_n = T_n(|f|^2) = [\tilde{a}_{r-c}]_{r,c=0}^{n-1}$, with $\tilde{a}_j$ being the Fourier coefficients of $|f|^2$, and setting $T_{n,\alpha}$ the α-Toeplitz generated by the same function $|f|^2$, it is immediate to observe

$$T_n\widehat{Z}_{n,\alpha} = \widehat{T}_{n,\alpha} \in \mathbb{C}^{n\times\mu_\alpha}, \qquad \text{with} \quad (\widehat{T}_{n,\alpha})_{r,c} = \tilde{a}_{r-\alpha c}, \tag{54}$$

for $r=0,\dots,n-1$ and $c=0,\dots,\mu_\alpha-1$. If we compute $\widehat{Z}_{n,\alpha}^*\widehat{T}_{n,\alpha}\in\mathbb{C}^{\mu_\alpha\times\mu_\alpha}$, where $Z_{n,\alpha}^*=[\delta_{c-\alpha r}]_{r,c=0}^{n-1}$ (δ_k defined as in (11)) and $\widehat{Z}_{n,\alpha}^*\in\mathbb{C}^{\mu_\alpha\times n}$ is the submatrix of $Z_{n,\alpha}^*$ obtained by considering only the μ_α first rows, for $r,c=0,\dots,\mu_\alpha-1$, we obtain

$$\begin{aligned}(\widehat{Z}_{n,\alpha}^*T_n(|f|^2)\widehat{Z}_{n,\alpha})_{r,c} &= (\widehat{Z}_{n,\alpha}^*\widehat{T}_{n,\alpha})_{r,c}\\ &= \sum_{\ell=0}^{n-1}(\widehat{Z}_{n,\alpha}^*)_{r,\ell}(\widehat{T}_{n,\alpha})_{\ell,c}\\ &\underset{(a)}{=} (\widehat{T}_{n,\alpha})_{\alpha r,c}\\ &\underset{\text{from (54)}}{=} \widehat{a}_{\alpha r-\alpha c},\end{aligned}$$

where (a) follows from the existence of a unique $\ell\in\{0,1,\dots,n-1\}$ such that $\ell-\alpha r\equiv 0 \pmod n$, that is, $\ell\equiv\alpha r \pmod n$, and, since $0\le\alpha r\le n-1$, we find $\ell=\alpha r$.

Therefore

$$\begin{aligned}\widehat{Z}_{n,\alpha}^*T_n(|f|^2)\widehat{Z}_{n,\alpha} &= [\tilde{a}_{\alpha r-\alpha c}]_{r,c=0}^{\mu_\alpha-1}\\ &= T_{\mu_\alpha}(\widehat{|f|^{(2)}}),\end{aligned}$$

where $\widehat{|f|^{(2)}}\in L^1(Q)$ is given by

$$\widehat{|f|^{(2)}}(x) = \frac{1}{\alpha}\sum_{j=0}^{\alpha-1}|f|^2\left(\frac{x+2\pi j}{\alpha}\right), \tag{55}$$

$$|f|^2(x) = \sum_{k=-\infty}^{+\infty}\tilde{a}_k e^{ikx}. \tag{56}$$

Proof. (of relation (55).) We denote by a_j the Fourier coefficients of $\widehat{|f|^{(2)}}$. We want to show that for $r,c=0,\dots,\mu_\alpha-1$, $\mathrm{a}_{r-c}=\tilde{a}_{\alpha r-\alpha c}$, where $\tilde{a}_k$ are the Fourier coefficients of $|f|^2$. From (4), (55) and (56), we have

$$\begin{aligned}\mathrm{a}_{r-c} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\alpha}\sum_{j=0}^{\alpha-1}\sum_{k=-\infty}^{+\infty}\tilde{a}_k e^{ik\left(\frac{x+2\pi j}{\alpha}\right)}e^{-i(r-c)x}dx\\ &= \frac{1}{2\pi\alpha}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{+\infty}\tilde{a}_k\left(\sum_{j=0}^{\alpha-1}e^{\frac{i2\pi kj}{\alpha}}\right)e^{\frac{ikx}{\alpha}}e^{-i(r-c)x}dx.\end{aligned}$$

Some remarks are in order:

- if k is a multiple of α, $k=\alpha t$ for some value of t, then we have that $\sum_{j=0}^{\alpha-1}e^{\frac{i2\pi kj}{\alpha}}=\sum_{j=0}^{\alpha-1}e^{\frac{i2\pi\alpha tj}{\alpha}}=\sum_{j=0}^{\alpha-1}e^{i2\pi tj}=\sum_{j=0}^{\alpha-1}1=\alpha$.
- if k is not a multiple of α, then $e^{\frac{i2\pi k}{\alpha}}\neq 1$ and therefore $\sum_{j=0}^{\alpha-1}e^{\frac{i2\pi kj}{\alpha}}=\sum_{j=0}^{\alpha-1}\left(e^{\frac{i2\pi k}{\alpha}}\right)^j$ is a finite geometric series whose sum is given by

$$\sum_{j=0}^{\alpha-1}\left(e^{\frac{i2\pi k}{\alpha}}\right)^j=\frac{1-e^{\frac{i2\pi k\alpha}{\alpha}}}{1-e^{\frac{i2\pi k}{\alpha}}}=\frac{1-e^{i2\pi k}}{1-e^{\frac{i2\pi k}{\alpha}}}=\frac{1-1}{1-e^{\frac{i2\pi k}{\alpha}}}=0.$$

Finally, taking into account the latter statements and recalling that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\ell x}dx = \left\{\begin{smallmatrix} 1 & \text{if } \ell=0 \\ 0 & \text{otherwise} \end{smallmatrix}\right.$, we find

$$\begin{aligned} \mathrm{a}_{r-c} &= \frac{1}{2\pi\alpha}\int_{-\pi}^{\pi}\sum_{t=-\infty}^{+\infty}\tilde{a}_{\alpha t}\alpha e^{\frac{i\alpha t x}{\alpha}} e^{-i(r-c)x}dx \\ &= \sum_{t=-\infty}^{+\infty}\tilde{a}_{\alpha t}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix(t-(r-c))}dx \\ &= \tilde{a}_{\alpha(r-c)}. \end{aligned}$$

□

In summary, from (51) we have

$$\widehat{T}^*_{n,\alpha}\widehat{T}_{n,\alpha} = T_{\mu_\alpha}(\widehat{|f|^{(2)}}) + \widehat{R}_{n,m,\alpha} + \widehat{N}_{n,m,\alpha},$$

with $\{T_{\mu_\alpha}(\widehat{|f|^{(2)}})\}_n \sim_\sigma (\widehat{|f|^{(2)}}, Q)$. We recall that, owing to (55), the relation $|f|^2 \in L^1(Q)$ implies $\widehat{|f|^{(2)}} \in L^1(Q)$. Consequently Proposition 4.1 implies that $\{\widehat{T}^*_{n,\alpha}\widehat{T}_{n,\alpha}\}_n \sim_\sigma (\widehat{|f|^{(2)}}, Q)$. Clearly $\widehat{|f|^{(2)}} \in L^1(Q)$ is equivalent to write $\sqrt{\widehat{|f|^{(2)}}} \in L^2(Q)$: therefore, from Lemma 4.1, we infer $\{\widehat{T}_{n,\alpha}\}_n \sim_\sigma (\sqrt{\widehat{|f|^{(2)}}}, Q)$.

Now, as mentioned at the beginning of this section, by Definition 2.1, we have

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} F\left(\sigma_j([\widehat{T}_{n,\alpha}|0])\right) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\mu_\alpha} F\left(\sigma_j([\widehat{T}_{n,\alpha}|0])\right) + \lim_{n\to\infty}\frac{1}{n}\sum_{j=\mu_\alpha+1}^{n} F(0) \\ &= \lim_{n\to\infty}\frac{\mu_\alpha}{n}\sum_{j=1}^{\mu_\alpha}\frac{F\left(\sigma_j([\widehat{T}_{n,\alpha}|0])\right)}{\mu_\alpha} + \lim_{n\to\infty}\frac{n-\mu_\alpha}{n}F(0) \\ &= \frac{1}{\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(\sqrt{\widehat{|f|^{(2)}}(x)}\right)dx + \left(1-\frac{1}{\alpha}\right)F(0), \end{aligned}$$

which results to be equivalent to the following distribution formula

$$\{[T_n\widehat{Z}_{n,\alpha}|0]\}_n \sim_\sigma (\theta, Q\times[0,1]), \tag{57}$$

where

$$\theta(x,t) = \begin{cases} \sqrt{\widehat{|f|^{(2)}}(x)} & \text{for } t \in \left[0, \frac{1}{\alpha}\right], \\ 0 & \text{for } t \in \left(\frac{1}{\alpha}, 1\right]. \end{cases} \tag{58}$$

4.2.2 Singular value distribution for the sequence $\{[0|\mathcal{T}_{n,\alpha}]\}_n$

In perfect analogy with the case of the matrix $[T_n\widehat{Z}_{n,\alpha}|0]$, we can observe that $\mathcal{T}_{n,\alpha} \in \mathbb{C}^{n\times(n-\mu_\alpha)}$ and $[0|\mathcal{T}_{n,\alpha}] \in \mathbb{C}^{n\times n}$. Therefore the matrix $[0|\mathcal{T}_{n,\alpha}]$ has μ_α singular values equal to zero and the remaining $n-\mu_\alpha$ equal to those of $\mathcal{T}_{n,\alpha}$. However, in this case we have additional difficulties with respect to the matrix $\widehat{T}_{n,\alpha} = T_n\widehat{Z}_{n,\alpha}$, because it is not always true that $\mathcal{T}_{n,\alpha}$ can be written as $T_n\mathcal{Z}_{n,\alpha}$, where $\mathcal{Z}_{n,\alpha}$ is the matrix obtained by considering the $n-\mu_\alpha$ last columns of $Z_{n,\alpha}$. Indeed, in $\mathcal{T}_{n,\alpha}$ there are Fourier coefficients with index, in modulus, greater than n: the Toeplitz matrix $T_n = [a_{r-c}]_{r,c=0}^{n-1}$ has coefficients a_j with j ranging from $1-n$ to $n-1$, while the α-Toeplitz

matrix $T_{n,\alpha} = [a_{r-\alpha c}]_{r,c=0}^{n-1}$ has a_{n-1} as coefficient of maximum index and $a_{-\alpha(n-1)}$ as coefficient of minimum index, and, if $\alpha \geq 2$, we have $-\alpha(n-1) < -(n-1)$.

Even if we take the Toeplitz matrix T_n, which has as its first column the first column of $\mathcal{T}_{n,\alpha}$ and the other generated according to the rule $(T_n)_{j,k} = a_{j-k}$, it is not always true that we can write $\mathcal{T}_{n,\alpha} = T_n P$ for a suitable submatrix P of a permutation matrix, indeed, if the matrix $T_n = [\beta_{r-c}]_{r,c=0}^{n-1}$ has as first column the first column of $\mathcal{T}_{n,\alpha}$, we find that $\beta_0 = (\mathcal{T}_{n,\alpha})_{0,0} = (T_{n,\alpha})_{0,\mu_\alpha} = a_{-\alpha\mu_\alpha}$. As a consequence, T_n has $\beta_{-(n-1)} = a_{-(n-1)-\alpha\mu_\alpha}$ as coefficient of minimum index, while $\mathcal{T}_{n,\alpha}$ has $a_{-\alpha(n-1)}$ as coefficient of minimum index. Therefore

$$\begin{aligned}
-(n-1)\alpha - (-(n-1) - \alpha\mu_\alpha) &= (1-\alpha)(n-1) + \alpha\mu_\alpha \qquad n \leq \alpha\mu_\alpha = \alpha\left\lceil \frac{n}{\alpha} \right\rceil \leq (n+\alpha-1) \\
&\leq (1-\alpha)(n-1) + (n+\alpha-1) \\
&= (1-\alpha)(n-1) + (n-1) + \alpha \\
&= (n-1)(1-\alpha+1) + \alpha \\
&= (2-\alpha)(n-1) + \alpha < 0 \qquad \text{for } \alpha > 2 \text{ and } n > 4.
\end{aligned}$$

Thus, if $\alpha > 2$ and $n > 4$ we have $-(n-1)\alpha < -(n-1) - \alpha\mu_\alpha$ and the coefficient of minimum index $a_{-\alpha(n-1)}$ of $\mathcal{T}_{n,\alpha}$ is not contained in the matrix T_n that has $a_{-(n-1)-\alpha\mu_\alpha}$ as coefficient of minimum index.

Then we proceed in another way: in the first column of $\mathcal{T}_{n,\alpha} \in \mathbb{C}^{n\times(n-\mu_\alpha)}$ (and consequently throughout the matrix) there are only coefficients with index < 0, indeed coefficient with the largest index of $\mathcal{T}_{n,\alpha}$ is $(\mathcal{T}_{n,\alpha})_{n-1,0} = (T_{n,\alpha})_{n-1,\mu_\alpha} = a_{n-1-\alpha\mu_\alpha}$ and $n-1-\alpha\mu_\alpha \leq n-1-n < 0$ and the coefficient with smallest index is $(\mathcal{T}_{n,\alpha})_{0,n-\mu_\alpha-1} = (T_{n,\alpha})_{0,n-\mu_\alpha-1+\mu_\alpha} = (T_{n,\alpha})_{0,n-1} = a_{-\alpha(n-1)}$. Consider therefore a Toeplitz matrix $T_{d_{n,\alpha}}$ of dimension $d_{n,\alpha}$ with $d_{n,\alpha} > \frac{\alpha(n-1)}{2} + 1$, defined in this way:

$$T_{d_{n,\alpha}} = \begin{bmatrix} a_{-d_{n,\alpha}+1} & a_{-d_{n,\alpha}} & a_{-d_{n,\alpha}-1} & \cdots & a_{-2d_{n,\alpha}+2} \\ a_{-d_{n,\alpha}+2} & a_{-d_{n,\alpha}+1} & \ddots & \ddots & a_{-2d_{n,\alpha}+3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-1} & a_{-2} & \ddots & \ddots & a_{-d_{n,\alpha}} \\ a_0 & a_{-1} & a_{-2} & \cdots & a_{-d_{n,\alpha}+1} \end{bmatrix} = \left[a_{r-c-d_{n,\alpha}+1}\right]_{r,c=0}^{d_{n,\alpha}-1}. \tag{59}$$

Since the coefficient with smallest index is $a_{-2d_{n,\alpha}+2}$, we find

$$-2d_{n,\alpha} + 2 < -2\left(\frac{\alpha(n-1)}{2} + 1\right) + 2 = -\alpha(n-1) - 2 + 2 = -\alpha(n-1).$$

As a consequence, we obtain that all the coefficients of $\mathcal{T}_{n,\alpha}$ are "contained" in the matrix $T_{d_{n,\alpha}}$. In particular, if

$$d_{n,\alpha} > (\alpha-1)(n-1) + 2,$$

(this condition ensures $d_{n,\alpha} > \frac{\alpha(n-1)}{2} + 1$, that all the subsequent inequalities are correct, and that the size of all the matrices involved are non-negative), then it can be shown that

$$\mathcal{T}_{n,\alpha} = [0_1|I_n|0_2]\, T_{d_{n,\alpha}} \mathcal{Z}_{d_{n,\alpha},\alpha}, \tag{60}$$

where $\mathcal{Z}_{d_{n,\alpha},\alpha} \in \mathbb{C}^{d_{n,\alpha}\times(n-\mu_\alpha)}$ is the matrix defined in (11), of dimension $d_{n,\alpha} \times d_{n,\alpha}$, by considering only the $n-\mu_\alpha$ first columns and $[0_1|I_n|0_2] \in \mathbb{C}^{n\times d_{n,\alpha}}$ is a block matrix with $0_1 \in \mathbb{C}^{n\times(d_{n,\alpha}-\alpha\mu_\alpha-1)}$ and $0_2 \in \mathbb{C}^{n\times(\alpha\mu_\alpha-n+1)}$.

Proof. (of relation (60).) First we observe that:

for $r = 0, 1, \ldots, n-1$ and $s = 0, 1, \ldots, n-\mu_\alpha - 1$ we have

$$(\mathcal{T}_{n,\alpha})_{r,s} = (T_{n,\alpha})_{r,s+\mu_\alpha} = a_{r-\alpha s-\alpha\mu_\alpha}; \tag{61}$$

for $r = 0, 1, \ldots, n-1$ and $s = 0, 1, \ldots, d_{n,\alpha} - 1$ we have

$$([0_1|I_n|0_2])_{r,s} = \begin{cases} 1 & \text{if } s = r + d_{n,\alpha} - \alpha\mu_\alpha - 1, \\ 0 & \text{otherwise;} \end{cases} \tag{62}$$

for $r, s = 0, 1, \ldots, d_{n,\alpha} - 1$ we have

$$(T_{d_{n,\alpha}})_{r,s} = a_{r-s-d_{n,\alpha}+1};$$

for $r = 0, 1, \ldots, d_{n,\alpha} - 1$ and $s = 0, 1, \ldots, n - \mu_\alpha - 1$, we have

$$(\mathcal{Z}_{d_{n,\alpha},\alpha})_{r,s} = \delta_{r-\alpha s}.$$

Since $T_{d_{n,\alpha}}\mathcal{Z}_{d_{n,\alpha},\alpha} \in \mathbb{C}^{d_{n,\alpha}\times(n-\mu_\alpha)}$, for $r = 0, 1, \ldots, d_{n,\alpha} - 1$ and $s = 0, 1, \ldots, n - \mu_\alpha - 1$, it holds

$$\begin{aligned} (T_{d_{n,\alpha}}\mathcal{Z}_{d_{n,\alpha},\alpha})_{r,s} &= \sum_{l=0}^{d_{n,\alpha}-1} (T_{d_{n,\alpha}})_{r,l}(\mathcal{Z}_{d_{n,\alpha},\alpha})_{l,s} \\ &= \sum_{l=0}^{d_{n,\alpha}-1} \delta_{l-\alpha s} a_{r-l-d_{n,\alpha}+1} \\ &\underset{\text{(a)}}{=} a_{r-\alpha s-d_{n,\alpha}+1}, \end{aligned} \tag{63}$$

where (a) follows from the existence of a unique $l \in \{0, 1, \ldots, d_{n,\alpha} - 1\}$ such that $l - \alpha s \equiv 0 \pmod{d_{n,\alpha}}$, that is, $l \equiv \alpha s \pmod{d_{n,\alpha}}$, and, since $0 \le \alpha s \le d_{n,\alpha} - 1$, we have $l = \alpha s$. Since $[0_1|I_n|0_2]\, T_{d_{n,\alpha}}\mathcal{Z}_{d_{n,\alpha},\alpha} \in \mathbb{C}^{n\times(n-\mu_\alpha)}$, for $r = 0, 1, \ldots, n-1$ and $s = 0, 1, \ldots, n - \mu_\alpha - 1$, we find

$$\begin{aligned} ([0_1|I_n|0_2]\, T_{d_{n,\alpha}}\mathcal{Z}_{d_{n,\alpha},\alpha})_{r,s} &= \sum_{l=0}^{d_{n,\alpha}-1} ([0_1|I_n|0_2])_{r,l}(T_{d_{n,\alpha}}\mathcal{Z}_{d_{n,\alpha},\alpha})_{l,s} \\ &\underset{\text{(d)}}{=} a_{r+d_{n,\alpha}-\alpha\mu_\alpha-1-\alpha s-d_{n,\alpha}+1} \\ &= a_{r-\alpha\mu_\alpha-\alpha s} \\ &\underset{\text{from (61)}}{=} (\mathcal{T}_{n,\alpha})_{r,s}, \end{aligned}$$

where (d) follows from (63), $(T_{d_{n,\alpha}}\mathcal{Z}_{d_{n,\alpha},\alpha})_{l,s} = a_{l-\alpha s-d_{n,\alpha}+1}$, and from the following fact: using (62), we find $([0_1|I_n|0_2])_{r,l} = 1$ if and only if $l = r + d_{n,\alpha} - \alpha\mu_\alpha - 1$. □

We can now observe immediately that the matrix $T_{d_{n,\alpha}}$ defined in (59) can be written as

$$T_{d_{n,\alpha}} = JH_{d_{n,\alpha}}, \tag{64}$$

where J is the "flip" matrix of dimension $d_{n,\alpha} \times d_{n,\alpha}$:

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix},$$

and $H_{d_{n,\alpha}}$ is the Hankel matrix of dimension $d_{n,\alpha} \times d_{n,\alpha}$:

$$H_{d_{n,\alpha}} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-d_{n,\alpha}+1} \\ a_{-1} & a_{-2} & \iddots & \iddots & a_{-d_{n,\alpha}} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ a_{-d_{n,\alpha}+2} & a_{-d_{n,\alpha}+1} & \iddots & \iddots & a_{-2d_{n,\alpha}+3} \\ a_{-d_{n,\alpha}+1} & a_{-d_{n,\alpha}} & a_{-d_{n,\alpha}-1} & \cdots & a_{-2d_{n,\alpha}+2} \end{bmatrix}.$$

If $f(x) \in L^1(Q)$, $Q = (-\pi, \pi)$, is the generating function of the Toeplitz matrix $T_n = T_n(f) = [a_{r-c}]_{r,c=0}^{n-1}$ in (39), where the k-th Fourier coefficient of f is a_k, then $f(-x) \in L^1(Q)$ is the generating function of the Hankel matrix $H_{d_{n,\alpha}} = [a_{-r-c}]_{r,c=0}^{d_{n,\alpha}-1}$; by invoking Theorem 6, page 161 of [7], the sequence of matrices $\{H_{d_{n,\alpha}}\}$ is distributed in the singular value sense as the zero function: $\{H_{d_{n,\alpha}}\} \sim_\sigma (0, Q)$. From Lemma 4.2, by (64), since J is a unitary matrix, we have $\{T_{d_{n,\alpha}}\} \sim_\sigma (0, Q)$ as well.

Consider the decomposition in (60):

$$\mathcal{T}_{n,\alpha} = [0_1|I_n|0_2]\, T_{d_{n,\alpha}} \mathcal{Z}_{d_{n,\alpha},\alpha} = Q_{d_{n,\alpha}} T_{d_{n,\alpha}} \mathcal{Z}_{d_{n,\alpha},\alpha}.$$

If we complete the matrices $Q_{d_{n,\alpha}} \in \mathbb{C}^{n \times d_{n,\alpha}}$ and $\mathcal{Z}_{d_{n,\alpha},\alpha} \in \mathbb{C}^{d_{n,\alpha} \times (n-\mu_\alpha)}$ by adding an appropriate number of zero rows and columns, respectively, in order to make it square

$$\begin{aligned} \mathbf{Q}_{d_{n,\alpha}} &= \left[\begin{array}{c} Q_{d_{n,\alpha}} \\ \hline 0 \end{array}\right] \in \mathbb{C}^{d_{n,\alpha} \times d_{n,\alpha}}, \\ \mathbf{Z}_{d_{n,\alpha},\alpha} &= \left[\begin{array}{c|c} \mathcal{Z}_{d_{n,\alpha},\alpha} & 0 \end{array}\right] \in \mathbb{C}^{d_{n,\alpha} \times d_{n,\alpha}}, \end{aligned}$$

then it is immediate to note that

$$\mathbf{Q}_{d_{n,\alpha}} T_{d_{n,\alpha}} \mathbf{Z}_{d_{n,\alpha},\alpha} = \left[\begin{array}{c|c} \mathcal{T}_{n,\alpha} & 0 \\ \hline 0 & 0 \end{array}\right] = \mathbf{T}_{n,\alpha} \in \mathbb{C}^{d_{n,\alpha} \times d_{n,\alpha}}.$$

From Lemma 4.3, since $\|\mathbf{Q}_{d_{n,\alpha}}\| = \|\mathbf{Z}_{d_{n,\alpha},\alpha}\| = 1$ (indeed they are both "incomplete" permutation matrices), and since $\{T_{d_{n,\alpha}}\} \sim_\sigma (0, Q)$, we infer that $\{\mathbf{T}_{n,\alpha}\} \sim_\sigma (0, Q)$.

Recall that $\mathbf{T}_{n,\alpha} \in \mathbb{C}^{d_{n,\alpha} \times d_{n,\alpha}}$ with $d_{n,\alpha} > (\alpha - 1)(n - 1) + 2$; then we can always choose $d_{n,\alpha}$ such that $\alpha n = d_{n,\alpha} > (\alpha - 1)(n - 1) + 2$ (if $n, \alpha \geq 2$). Now, since $\{\mathbf{T}_{n,\alpha}\} \sim_\sigma (0, Q)$, it holds that the sequence $\{\mathbf{T}_{n,\alpha}\}$ is weakly clustered at zero in the singular value sense, i.e., $\forall \epsilon > 0$,

$$\sharp\{j : \sigma_j(\mathbf{T}_{n,\alpha}) > \epsilon\} = o(d_{n,\alpha}) = o(\alpha n) = o(n). \tag{65}$$

The matrix $\mathbf{T}_{n,\alpha}$ is a block matrix that can be written as

$$\mathbf{T}_{n,\alpha} = \left[\begin{array}{c|c} \mathcal{T}_{n,\alpha} & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} [\mathcal{T}_{n,\alpha}|0] & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $\mathcal{T}_{n,\alpha} \in \mathbb{C}^{n \times (n-\mu_\alpha)}$ and $[\mathcal{T}_{n,\alpha}|0] \in \mathbb{C}^{n \times n}$. By the singular value decomposition we obtain

$$\mathbf{T}_{n,\alpha} = \left[\begin{array}{c|c} [\mathcal{T}_{n,\alpha}|0] & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} U_1 \Sigma_1 V_1^* & 0 \\ \hline 0 & U_2 0 V_2^* \end{array}\right] = \left[\begin{array}{c|c} U_1 & 0 \\ \hline 0 & U_2 \end{array}\right] \left[\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array}\right]^*,$$

that is, the singular values of $\mathbf{T}_{n,\alpha}$ that are different from zero are the singular values of $[\mathcal{T}_{n,\alpha}|0] \in \mathbb{C}^{n \times n}$. Thus (65) can be written as follows: $\forall \epsilon > 0$,

$$\sharp\{j : \sigma_j([\mathcal{T}_{n,\alpha}|0]) > \epsilon\} = o(d_{n,\alpha}) = o(\alpha n) = o(n).$$

The latter relation means that the sequence $\{[\mathcal{T}_{n,\alpha}|0]\}_n$ is weakly clustered at zero in the singular value sense, and hence $\{[\mathcal{T}_{n,\alpha}|0]\}_n \sim_\sigma (0, Q)$. If we now consider the matrix

$$\hat{G} = \left[\begin{array}{c|c} 0 & I_{n-\mu_\alpha} \\ \hline 0 & 0 \end{array}\right] \in \mathbb{C}^{n\times n},$$

where $I_{n-\mu_\alpha}$ is the identity matrix of dimension $(n-\mu_\alpha)\times(n-\mu_\alpha)$, then $[\mathcal{T}_{n,\alpha}|0]\hat{G} = [0|\mathcal{T}_{n,\alpha}]$, and since $\|\hat{G}\| = 1$ and $\{[\mathcal{T}_{n,\alpha}|0]\}_n \sim_\sigma (0, Q)$, from Lemma 4.3 we find

$$\{[0|\mathcal{T}_{n,\alpha}]\}_n \sim_\sigma (0, Q). \tag{66}$$

In conclusion: from the relations (49), (57) and (66), using Proposition 4.2 with $G = Q \times [0, 1]$, we obtain that

$$\{T_{n,\alpha}\}_n \sim_\sigma (\theta, Q \times [0, 1]),$$

where θ is defined in (58). Notice that for $\alpha = 1$ the symbol $\theta(x, t)$ coincides with $|f|(x)$ on the extended domain $Q \times [0, 1]$. Hence the Szegö-Tilli-Tyrtyshnikov-Zamarashkin result is found as a particular case. Indeed $\theta(x, t) = |f|(x)$ does not depend on t and therefore this additional variable can be suppressed i.e. $\{T_{n,\alpha}\}_n \sim_\sigma (f, Q)$ with $T_{n,\alpha} = T_n(f)$. The fact that the distribution formula is not unique should not surprise since this phenomenon is inherent to the measure theory because any measure-preserving exchange function is a distribution function if one representative of the class is.

5 Some remarks on multigrid methods

In the design of multigrid methods for large positive definite linear systems one of the key points is to maintain the structure (if any) of the original matrix in the lower levels. This means that at every recursion level the new projected linear system should retain the main properties of the original matrix (e.g. bandedness, the same level of conditioning, the same algebra/Toeplitz/graph structure etc.). Here for the sake of simplicity the example that has to be considered is the one-level circulant case. Following [1, 21], if $A_n = C_n$ is a positive circulant matrix of size n with n power of 2, then the projected matrix A_k with $k = n/2$ is defined as

$$A_k = \widetilde{Z}_{n,2}^T P_n^* A_n P_n \widetilde{Z}_{n,2}, \tag{67}$$

where P_n is an additional circulant matrix. It is worth noticing that the structure is kept since for every circulant P_n the matrix A_k is a circulant matrix of size $k = n/2$. The features of the specific P_n have to be designed in such a way that the convergence speed of the related multigrid is as high as possible (see [9, 1] for a general strategy). We observe that the eigenvalues of A_k are given by

$$\frac{1}{2}\sum_{l=0}^{1} g\left(\frac{x_j + 2\pi l}{2}\right), \quad x_j = \frac{2\pi j}{k}, \quad j = 0, 1, \ldots, k-1, \;\; k = n/2, \tag{68}$$

where g is the polynomial associated with the circulant matrix $P_n^* A_n P_n$ in the sense of Subsection 3.3. Therefore the singular values of $(P_n^* A_n P_n)^{1/2} \widetilde{Z}_{n,2}$ are given by

$$\frac{1}{\sqrt{2}}\sqrt{\sum_{l=0}^{1} g\left(\frac{x_j + 2\pi l}{2}\right)}, \quad x_j = \frac{2\pi j}{k}, \quad j = 0, 1, \ldots, k-1, \;\; k = n/2. \tag{69}$$

Notice that the latter formula is a special instance of (37) for $|p|^2 = g$ (g is necessarily nonnegative since it can be written a $|q|^2 f$ where q is the polynomial associated with P_n and f the nonnegative polynomial associated with A_n), for $\alpha = 2$ and n even number so that $(n, 2) = 2$. Therefore, according to (37), the numbers in (69) identify the nontrivial singular values of the 2-circulant matrix $(P_n^* A_n P_n)^{1/2} Z_{n,2}$ up to a scaling factor. In other words α-circulant matrices arise naturally in the design of fast multigrid solvers for circulant linear systems and, along the same lines, α-Toeplitz matrices arise naturally in the design of fast multigrid solvers for Toeplitz linear systems; see [9, 1, 15].

Conversely, we now can see clearly that formula (37) furnishes a wide generalization of the spectral analysis of the projected matrices, by allowing a higher degree of freedom: we can choose n divisible by α with $\alpha \neq 2$, we can choose n not divisible by α. Such a degree of freedom is not just academic, but could be exploited for devising optimally convergent multigrid solvers also in critical cases emphasized e.g. in [1, 15]. In particular, if x_0 is an isolated zero of f (the nonnegative polynomial related to $A_n = C_n$) and also $\pi + x_0$ is a zero for the same function, then due to special symmetries, the associated multigrid (or even two-grid) method cannot be optimal. In other words, for reaching a preassigned accuracy, we cannot expect a number of iterations independent of the order n. However these pathological symmetries are due to the choice of $\alpha = 2$, so that a choice of a projector as $P_n \widetilde{Z}_{n,\alpha}$ for a different $\alpha \neq 2$ and a different n could completely overcome the latter drawback.

6 Generalizations

First of all we observe that the requirement that the symbol f is square integrable can be removed. In [14] it is proven that the singular value distribution of $\{T_n(f)T_n(g)\}_n$ is given by $h = fg$ with f, g being just Lebesgue integrable and with h that is only measurable and therefore may fail to be Lebesgue integrable. This fact is sufficient for extending the proof of the relation $\{T_{n,\alpha}\}_n \sim_\sigma (\theta, Q \times [0, 1])$ to the case where $\theta(x, t)$ is defined as in (58) with the original symbol $f \in L^1$.

Now we consider the general multilevel case. When α is a positive vector, we have

$$\{T_{n,\alpha}\}_n \sim_\sigma (\theta, Q^d \times [0, 1]^d), \tag{70}$$

where

$$\theta(x, t) = \begin{cases} \sqrt{\widehat{|f|^{(2)}}(x)} & \text{for } t \in \left[\underline{0}, \frac{1}{\alpha}\right], \\ 0 & \text{for } t \in \left(\frac{1}{\alpha}, e\right], \end{cases} \tag{71}$$

with

$$\widehat{|f|^{(2)}}(x) = \frac{1}{\hat{\alpha}} \sum_{j=\underline{0}}^{\alpha-e} |f|^2 \left(\frac{x + 2\pi j}{\alpha}\right), \tag{72}$$

and where all the arguments are modulus 2π and all the operations are intended componentwise that is $t \in \left[\underline{0}, \frac{1}{\alpha}\right]$ means that $t_k \in [0, 1/\alpha_k]$, $k = 1, \dots, d$, $t \in \left(\frac{1}{\alpha}, e\right]$ means that $t_k \in (1/\alpha_k, 1]$, $k = 1, \dots, d$, the writing $\frac{x+2\pi j}{\alpha}$ defines the d-dimensional vector whose k-th component is $(x_j + 2\pi j_k)/\alpha_k$, $k = 1, \dots, d$, and $\hat{\alpha} = \alpha_1 \alpha_2 \cdots \alpha_d$.

6.0.3 Examples of α-circulant and α-Toeplitz matrices when some of the entries of α vanish

We start this subsection with a brief digression on multilevel matrices. A d-level matrix A of dimension $\hat{n} \times \hat{n}$ with $n = (n_1, n_2, \dots, n_d)$ and $\hat{n} = n_1 n_2 \cdots n_d$ can be viewed as a matrix of

dimension $n_1 \times n_1$ in which each element is a block of dimension $n_2 n_3 \cdots n_d \times n_2 n_3 \cdots n_d$; in turn, each block of dimension $n_2 n_3 \cdots n_d \times n_2 n_3 \cdots n_d$ can be viewed as a matrix of dimension $n_2 \times n_2$ in which each element is a block of dimension $n_3 n_4 \cdots n_d \times n_3 n_4 \cdots n_d$, and so on. So we can say that n_1 is the most "outer" dimension of the matrix A and n_d is the most "inner" dimension. If we multiply by an appropriate permutation matrix P the d-level matrix A, we can exchange the "order of dimensions" of A, namely $P^T AP$ becomes a matrix again of dimension $\hat{n} \times \hat{n}$ but with $n = (n_{p(1)}, n_{p(2)}, \ldots, n_{p(d)})$ and $\hat{n} = n_{p(1)} n_{p(2)} \cdots n_{p(d)} = n_1 n_2 \cdots n_d$ (where p is a permutation of d elements) and $n_{p(1)}$ is the most "outer" dimension of the matrix A and $n_{p(d)}$ is the most "inner" dimension.

This trick helps us to understand what happens to the singular values of α-circulant and α-Toeplitz d-level matrices, especially when some of the entries of the vector α are zero; indeed, as we observed in Subsection 2.1.2, if $\alpha = \underline{0}$, the d-level α-circulant (or α-Toeplitz) matrix A is a block matrix with constant blocks on each row, so if we order the vector α (which has some components equal to zero) so that the components equal to zero are in the top positions, $\alpha = (0, \ldots, 0, \alpha_k, \ldots, \alpha_d)$, the matrix $P^T AP$ (where P is the permutation matrix associated with p) becomes a block matrix with constant blocks on each row and with blocks of dimension $n_k \cdots n_d \times n_k \cdots n_d$; with this "new" structure, formulas (8) and (9) are even more intuitively understandable, as we shall see later in the examples.

Lemma 6.1. *Let A be a 2-level Toeplitz matrix of dimension $\hat{n} \times \hat{n}$ with $n = (n_1, n_2)$ and $\hat{n} = n_1 n_2$,*

$$A = \left[\left[a_{(j_1 - k_1, j_2 - k_2)} \right]_{j_2, k_2 = 0}^{n_2 - 1} \right]_{j_1, k_1 = 0}^{n_1 - 1}.$$

There exists a permutation matrix P such that

$$P^T AP = \left[\left[a_{(j_1 - k_1, j_2 - k_2)} \right]_{j_1, k_1 = 0}^{n_1 - 1} \right]_{j_2, k_2 = 0}^{n_2 - 1}.$$

Example: Let $n = (n_1, n_2) = (2, 3)$ and consider the 2-level Toeplitz matrix A of dimension 6×6

$$A = \left[\begin{array}{ccc|ccc} a_{(0,0)} & a_{(0,-1)} & a_{(0,-2)} & a_{(-1,0)} & a_{(-1,-1)} & a_{(-1,-2)} \\ a_{(0,1)} & a_{(0,0)} & a_{(0,-1)} & a_{(-1,1)} & a_{(-1,0)} & a_{(-1,-1)} \\ a_{(0,2)} & a_{(0,1)} & a_{(0,0)} & a_{(-1,2)} & a_{(-1,1)} & a_{(-1,0)} \\ \hline a_{(1,0)} & a_{(1,-1)} & a_{(1,-2)} & a_{(0,0)} & a_{(0,-1)} & a_{(0,-2)} \\ a_{(1,1)} & a_{(1,0)} & a_{(1,-1)} & a_{(0,1)} & a_{(0,0)} & a_{(0,-1)} \\ a_{(1,2)} & a_{(1,1)} & a_{(1,0)} & a_{(0,2)} & a_{(0,1)} & a_{(0,0)} \end{array} \right].$$

This matrix can be viewed as a matrix of dimension 2×2 in which each element is a block of dimension 3×3. If we take the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then it is plain to see that

$$P^T AP = \left[\begin{array}{cc|cc|cc} a_{(0,0)} & a_{(-1,0)} & a_{(0,-1)} & a_{(-1,-1)} & a_{(0,-2)} & a_{(-1,-2)} \\ a_{(1,0)} & a_{(0,0)} & a_{(1,-1)} & a_{(0,-1)} & a_{(1,-2)} & a_{(0,-2)} \\ \hline a_{(0,1)} & a_{(-1,1)} & a_{(0,0)} & a_{(-1,0)} & a_{(0,-1)} & a_{(-1,-1)} \\ a_{(1,1)} & a_{(0,1)} & a_{(1,0)} & a_{(0,0)} & a_{(1,-1)} & a_{(0,-1)} \\ \hline a_{(0,2)} & a_{(-1,2)} & a_{(0,1)} & a_{(-1,1)} & a_{(0,0)} & a_{(-1,0)} \\ a_{(1,2)} & a_{(0,2)} & a_{(1,1)} & a_{(0,1)} & a_{(1,0)} & a_{(0,0)} \end{array}\right],$$

and now $P^T AP$ can be naturally viewed as a matrix of dimension 3×3 in which each element is a block of dimension 2×2.

Corollary 6.1. *Let A be a d-level Toeplitz matrix of dimension $\hat{n} \times \hat{n}$ with $n = (n_1, n_2, \ldots, n_d)$ and $\hat{n} = n_1 n_2 \cdots n_d$,*

$$A = \left[\left[\cdots \left[a_{(j_1-k_1, j_2-k_2, \ldots, j_d-k_d)}\right]_{j_d,k_d=0}^{n_d-1} \cdots\right]_{j_2,k_2=0}^{n_2-1}\right]_{j_1,k_1=0}^{n_1-1}.$$

For every permutation p of d elements, there exists a permutation matrix P such that

$$P^T AP = \left[\left[\cdots \left[a_{(j_1-k_1, j_2-k_2, \ldots, j_d-k_d)}\right]_{j_{p(d)},k_{p(d)}=0}^{n_{p(d)}-1} \cdots\right]_{j_{p(2)},k_{p(2)}=0}^{n_{p(2)}-1}\right]_{j_{p(1)},k_{p(1)}=0}^{n_{p(1)}-1}.$$

Remark 6.1. *Lemma 6.1 and Corollary 6.1 also apply to d-level α-circulant and α-Toeplitz matrices.*

Now, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be a d-dimensional vector of nonnegative integers and $t = \sharp\{j : \alpha_j = 0\}$ be the number of zero entries of α. If we take a permutation p of d elements such that $\alpha_{p(1)} = \alpha_{p(2)} = \ldots = \alpha_{p(t)} = 0$, (that is, p is a permutation that moves all the zero components of the vector α in the top positions), then it is easy to prove that formulas (8) and (9) remain the same for the matrix $P^T AP$ (where P is the permutation matrix associated with p) but with $n[0] = (n_{p(1)}, n_{p(2)}, \ldots, n_{p(t)})$ and where C_j and T_j are a d^+-level α^+-circulant and α^+-Toeplitz matrix, respectively, with $\alpha^+ = (\alpha_{p(t+1)}, \alpha_{p(t+2)}, \ldots, \alpha_{p(d)})$, of partial sizes $n[>0] = (n_{p(t+1)}, n_{p(t+2)}, \ldots, n_{p(d)})$, and whose expressions are

$$\begin{aligned} C_j &= \left[\left[\cdots \left[a_{(r-\alpha\circ s) \bmod n}\right]_{r_{p(d)},s_{p(d)}=0}^{n_{p(d)}-1} \cdots\right]_{r_{p(t+2)},s_{p(t+2)}=0}^{n_{p(t+2)}-1}\right]_{r_{p(t+1)},s_{p(t+1)}=0}^{n_{p(t+1)}-1}, \\ T_j &= \left[\left[\cdots \left[a_{(r-\alpha\circ s)}\right]_{r_{p(d)},s_{p(d)}=0}^{n_{p(d)}-1} \cdots\right]_{r_{p(t+2)},s_{p(t+2)}=0}^{n_{p(t+2)}-1}\right]_{r_{p(t+1)},s_{p(t+1)}=0}^{n_{p(t+1)}-1}, \end{aligned}$$

with $(r_{p(1)}, r_{p(2)}, \ldots, r_{p(t)}) = j$. Obviously $\mathrm{Sgval}(A) = \mathrm{Sgval}(P^T AP)$.

We recall that if B is a matrix of size $n \times n$ positive semidefinite, that is $B^* = B$ and $x^* Bx \geq 0$ $\forall x \neq 0$, then $\mathrm{Eig}(B) = \mathrm{Sgval}(B)$. Moreover, if $B = U\Sigma U^*$ is a SVD for B (which coincides with the Schur decomposition of B) with $\Sigma = \underset{j=1,\ldots,n}{\mathrm{diag}}(\sigma_j)$, then

$$B^{1/2} = U\Sigma^{1/2}U^*, \tag{73}$$

where $\Sigma^{1/2} = \underset{j=1,\ldots,n}{\mathrm{diag}}(\sqrt{\sigma_j})$.

We proceed with two detailed examples: a 3-level α-circulant matrix with $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (1, 2, 0)$, and a 3-level α-Toeplitz with $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (0, 1, 2)$, which helps us to understand what happens if the vector α is not strictly positive. Finally we will propose the explicit calculation of the singular values of a d-level α-circulant matrix in the particular case where the vector α has only one component different from zero.

Example: Consider a 3-level α-circulant matrix A where $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (1, 2, 0)$

$$\begin{aligned} A &= \left[\left[\left[a_{((r_1 - 1\cdot s_1) \bmod n_1, (r_2 - 2\cdot s_2) \bmod n_2, (r_3 - 0\cdot s_3) \bmod n_3)}\right]_{r_3,s_3=0}^{n_3-1}\right]_{r_2,s_2=0}^{n_2-1}\right]_{r_1,s_1=0}^{n_1-1} \\ &= \left[\left[\left[a_{((r_1 - s_1) \bmod n_1, (r_2 - 2s_2) \bmod n_2, r_3)}\right]_{r_3=0}^{n_3-1}\right]_{r_2,s_2=0}^{n_2-1}\right]_{r_1,s_1=0}^{n_1-1}. \end{aligned}$$

If we choose a permutation p of 3 elements such that

$$\begin{aligned} &(p(1), p(2), p(3)) = (3, 2, 1), \\ &(\alpha_{p(1)}, \alpha_{p(2)}, \alpha_{p(3)}) = (0, 2, 1), \\ &(n_{p(1)}, n_{p(2)}, n_{p(3)}) = (n_3, n_2, n_1), \end{aligned}$$

and if we take the permutation matrix P related to p, then

$$P^T A P \equiv \hat{A} = \left[\left[\left[a_{((r_1 - s_1) \bmod n_1, (r_2 - 2s_2) \bmod n_2, r_3)}\right]_{r_1,s_1=0}^{n_1-1}\right]_{r_2,s_2=0}^{n_2-1}\right]_{r_3=0}^{n_3-1}.$$

Now, for $r_3 = 0, 1, ..., n_3 - 1$, let us set

$$C_{r_3} = \left[\left[a_{((r_1 - s_1) \bmod n_1, (r_2 - 2s_2) \bmod n_2, r_3)}\right]_{r_1,s_1=0}^{n_1-1}\right]_{r_2,s_2=0}^{n_2-1}.$$

As a consequence, C_{r_3} is a 2-level α^+-circulant matrix with $\alpha^+ = (2, 1)$ and of partial sizes $n[>0] = (n_2, n_1)$ and the matrix $\hat{A}$ can be rewritten as

$$\hat{A} = \begin{bmatrix} C_0 & C_0 & \cdots & C_0 \\ C_1 & C_1 & \cdots & C_1 \\ \vdots & \vdots & \vdots & \vdots \\ C_{n_3-1} & C_{n_3-1} & \cdots & C_{n_3-1} \end{bmatrix},$$

and this is a block matrix with constant blocks on each row. From formula (1), the singular

values of $\hat{A}$ are the square root of the eigenvalues of $\hat{A}^*\hat{A}$:

$$
\begin{aligned}
\hat{A}^*\hat{A} &= \begin{bmatrix} C_0^* & C_1^* & \cdots & C_{n_3-1}^* \\ C_0^* & C_1^* & \cdots & C_{n_3-1}^* \\ \vdots & \vdots & \vdots & \vdots \\ C_0^* & C_1^* & \cdots & C_{n_3-1}^* \end{bmatrix} \begin{bmatrix} C_0 & C_0 & \cdots & C_0 \\ C_1 & C_1 & \cdots & C_1 \\ \vdots & \vdots & \vdots & \vdots \\ C_{n_3-1} & C_{n_3-1} & \cdots & C_{n_3-1} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=0}^{n_3-1} C_j^* C_j & \sum_{j=0}^{n_3-1} C_j^* C_j & \cdots & \sum_{j=0}^{n_3-1} C_j^* C_j \\ \sum_{j=0}^{n_3-1} C_j^* C_j & \sum_{j=0}^{n_3-1} C_j^* C_j & \cdots & \sum_{j=0}^{n_3-1} C_j^* C_j \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=0}^{n_3-1} C_j^* C_j & \sum_{j=0}^{n_3-1} C_j^* C_j & \cdots & \sum_{j=0}^{n_3-1} C_j^* C_j \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{n_3 \text{ times}} \otimes \sum_{j=0}^{n_3-1} C_j^* C_j \\
&= J_{n_3} \otimes \sum_{j=0}^{n_3-1} C_j^* C_j.
\end{aligned}
$$

Therefore

$$
\operatorname{Eig}(\hat{A}^*\hat{A}) = \operatorname{Eig}\left(J_{n_3} \otimes \sum_{j=0}^{n_3-1} C_j^* C_j \right), \tag{74}
$$

where

$$
\operatorname{Eig}(J_{n_3}) = \{0, n_3\}, \tag{75}
$$

because J_{n_3} is a matrix of rank 1, so it has all eigenvalues equal to zero except one eigenvalue equal to $\operatorname{tr}(J_{n_3}) = n_3$ (tr is the trace of a matrix). If we put

$$
\lambda_k = \lambda_k \left(\sum_{j=0}^{n_3-1} C_j^* C_j \right), \qquad k = 0, \ldots, n_1 n_2 - 1,
$$

by exploiting basic properties of the tensor product and taking into consideration (74) and (75) we find

$$
\begin{aligned}
\lambda_k(\hat{A}^*\hat{A}) &= n_3 \lambda_k, \qquad k = 0, \ldots, n_1 n_2 - 1, && (76) \\
\lambda_k(\hat{A}^*\hat{A}) &= 0, \qquad k = n_1 n_2, \ldots, n_1 n_2 n_3 - 1. && (77)
\end{aligned}
$$

From (76), (77) and (1), and recalling that $\operatorname{Sgval}(\hat{A}) = \operatorname{Sgval}(A)$, one obtains that the singular values of A are given by

$$
\begin{aligned}
\sigma_k(A) &= \sqrt{n_3 \lambda_k}, \qquad k = 0, \ldots, n_1 n_2 - 1, \\
\sigma_k(A) &= 0, \qquad k = n_1 n_2, \ldots, n_1 n_2 n_3 - 1,
\end{aligned}
$$

and, since $\sum_{j=0}^{n_3-1} C_j^* C_j$ is a positive semidefinite matrix, from (73) we can write

$$\begin{aligned}\sigma_k(A) &= \sqrt{n_3}\widetilde{\sigma}_k, \qquad k = 0, \ldots, n_1 n_2 - 1, \\ \sigma_k(A) &= 0, \qquad k = n_1 n_2, \ldots, n_1 n_2 n_3 - 1,\end{aligned}$$

where $\widetilde{\sigma}_k$ are the singular values of $\left(\sum_{j=0}^{n_3-1} C_j^* C_j\right)^{1/2}$.

Regarding the distribution in the sense of singular values, let $F \in C_0(\mathbb{R}_0^+)$, continuous function over $\mathbb{R}_0^+$ with bounded support, then there exists $a \in \mathbb{R}^+$ such that

$$|F(x)| \leq a \ \ \forall x \in \mathbb{R}_0^+. \tag{78}$$

From formula (2) we have

$$\begin{aligned}\Sigma_\sigma(F, A_n) &= \frac{1}{n_1 n_2 n_3} \sum_{k=0}^{n_1 n_2 n_3 - 1} F(\sqrt{n_3}\widetilde{\sigma}_k) \\ &= \frac{n_1 n_2 (n_3 - 1) F(0)}{n_1 n_2 n_3} + \frac{1}{n_1 n_2 n_3} \sum_{k=0}^{n_1 n_2 - 1} F(\sqrt{n_3}\widetilde{\sigma}_k) \\ &= \left(1 - \frac{1}{n_3}\right) F(0) + \frac{1}{n_1 n_2 n_3} \sum_{k=0}^{n_1 n_2 - 1} F(\sqrt{n_3}\widetilde{\sigma}_k).\end{aligned}$$

According to (78), we find

$$-a n_1 n_2 \leq \sum_{k=0}^{n_1 n_2 - 1} F(\sqrt{n_3}\widetilde{\sigma}_k) \leq a n_1 n_2.$$

Therefore

$$-\frac{a}{n_3} \leq \frac{1}{n_1 n_2 n_3} \sum_{k=0}^{n_1 n_2 - 1} F(\sqrt{n_3}\widetilde{\sigma}_k) \leq \frac{a}{n_3},$$

so that

$$\left(1 - \frac{1}{n_3}\right) F(0) - \frac{a}{n_3} \leq \Sigma_\sigma(F, A_n) \leq \left(1 - \frac{1}{n_3}\right) F(0) + \frac{a}{n_3}.$$

Now, recalling that the writing $n \to \infty$ means $\min_{1 \leq j \leq 3} n_j \to \infty$, we obtain

$$F(0) \leq \lim_{n \to \infty} \Sigma_\sigma(F, A_n) \leq F(0),$$

which implies

$$\lim_{n \to \infty} \Sigma_\sigma(F, A_n) = F(0).$$

Whence

$$\{A_n\} \sim_\sigma (0, G),$$

for any domain G satisfying the requirements of Definition 2.1.

Example: Consider a 3-level α-Toeplitz matrix A where $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (0, 1, 2)$

$$\begin{aligned} A &= \left[\left[\left[a_{(r_1-0\cdot s_1, r_2-1\cdot s_2, r_3-2\cdot s_3)}\right]_{r_3,s_3=0}^{n_3-1}\right]_{r_2,s_2=0}^{n_2-1}\right]_{r_1,s_1=0}^{n_1-1} \\ &= \left[\left[\left[a_{(r_1, r_2-s_2, r_3-2s_3)}\right]_{r_3,s_3=0}^{n_3-1}\right]_{r_2,s_2=0}^{n_2-1}\right]_{r_1=0}^{n_1-1}. \end{aligned}$$

The procedure is the same as in the previous example of an α-circulant matrix, but in this case we do not need to permute the vector α since the only component equal to zero is already in first position. For $r_1 = 0, 1, ..., n_1 - 1$, let us set

$$T_{r_1} = \left[\left[a_{(r_1, r_2-s_2, r_3-2s_3)}\right]_{r_3,s_3=0}^{n_3-1}\right]_{r_2,s_2=0}^{n_2-1},$$

then T_{r_1} is a 2-level α^+-Toeplitz matrix with $\alpha^+ = (1, 2)$ and of partial sizes $n[> 0] = (n_2, n_3)$ and

$$A = \begin{bmatrix} T_0 & T_0 & \cdots & T_0 \\ T_1 & T_1 & \cdots & T_1 \\ \vdots & \vdots & \vdots & \vdots \\ T_{n_1-1} & T_{n_1-1} & \cdots & T_{n_1-1} \end{bmatrix}.$$

The latter is a block matrix with constant blocks on each row. From formula (1), the singular values of A are the square root of the eigenvalues of A^*A:

$$\begin{aligned} A^*A &= \begin{bmatrix} T_0^* & T_1^* & \cdots & T_{n_1-1}^* \\ T_0^* & T_1^* & \cdots & T_{n_1-1}^* \\ \vdots & \vdots & \vdots & \vdots \\ T_0^* & T_1^* & \cdots & T_{n_1-1}^* \end{bmatrix} \begin{bmatrix} T_0 & T_0 & \cdots & T_0 \\ T_1 & T_1 & \cdots & T_1 \\ \vdots & \vdots & \vdots & \vdots \\ T_{n_1-1} & T_{n_1-1} & \cdots & T_{n_1-1} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=0}^{n_1-1} T_j^*T_j & \sum_{j=0}^{n_1-1} T_j^*T_j & \cdots & \sum_{j=0}^{n_1-1} T_j^*T_j \\ \sum_{j=0}^{n_1-1} T_j^*T_j & \sum_{j=0}^{n_1-1} T_j^*T_j & \cdots & \sum_{j=0}^{n_1-1} T_j^*T_j \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=0}^{n_1-1} T_j^*T_j & \sum_{j=0}^{n_1-1} T_j^*T_j & \cdots & \sum_{j=0}^{n_1-1} T_j^*T_j \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{n_1 \text{ times}} \otimes \sum_{j=0}^{n_1-1} T_j^*T_j \\ &= J_{n_1} \otimes \sum_{j=0}^{n_1-1} T_j^*T_j. \end{aligned}$$

Therefore

$$\mathrm{Eig}(A^*A) = \mathrm{Eig}\left(J_{n_1} \otimes \sum_{j=0}^{n_1-1} T_j^*T_j\right), \tag{79}$$

where

$$\mathrm{Eig}(J_{n_1}) = \{0, n_1\}, \tag{80}$$

because J_{n_1} is a matrix of rank 1, so it has all eigenvalues equal to zero except one eigenvalue equal to $\mathrm{tr}(J_{n_1}) = n_1$ (tr is the trace of a matrix). If we put

$$\lambda_k = \lambda_k\left(\sum_{j=0}^{n_1-1} T_j^* T_j\right), \qquad k = 0, \ldots, n_3 n_2 - 1,$$

by exploiting basic properties of the tensor product and taking into consideration (79) and (80) we find

$$\begin{aligned} \lambda_k(A^*A) &= n_1 \lambda_k, \qquad k = 0, \ldots, n_3 n_2 - 1, && (81)\\ \lambda_k(A^*A) &= 0, \qquad k = n_3 n_2, \ldots, n_3 n_2 n_1 - 1. && (82) \end{aligned}$$

From (81), (82) and (1), one obtains that the singular values of A are given by

$$\begin{aligned} \sigma_k(A) &= \sqrt{n_1 \lambda_k}, \qquad k = 0, \ldots, n_3 n_2 - 1, && (83)\\ \sigma_k(A) &= 0, \qquad k = n_3 n_2, \ldots, n_3 n_2 n_1 - 1. && (84) \end{aligned}$$

and, since $\sum_{j=0}^{n_1-1} T_j^* T_j$ is a positive semidefinite matrix, from (73) we can write

$$\begin{aligned} \sigma_k(A) &= \sqrt{n_1} \widetilde{\sigma}_k, \qquad k = 0, \ldots, n_3 n_2 - 1,\\ \sigma_k(A) &= 0, \qquad k = n_3 n_2, \ldots, n_3 n_2 n_1 - 1, \end{aligned}$$

where $\widetilde{\sigma}_k$ denotes the generic singular value of $\left(\sum_{j=0}^{n_1-1} T_j^* T_j\right)^{1/2}$.

Regarding the distribution in the sense of singular values, by invoking exactly the same argument as in the above example for α-circulant matrix, we deduce that

$$\{A_n\} \sim_\sigma (0, G),$$

for any domain G satisfying the requirements of Definition 2.1.

Example: Let us see what happens when the vector α has only one component different from zero. Let $n = (n_1, n_2, \ldots, n_d)$ and $\alpha = (0, \ldots, 0, \alpha_k, 0, \ldots, 0)$, $\alpha_k > 0$; in this case we can give an explicit formula for the singular values of the d-level α-circulant matrix. For convenience and without loss of generality we take $\alpha = (0, \ldots, 0, \alpha_d)$ (with all zero components in top positions, otherwise we use a permutation). From 2.1.3, the singular values of $A_n = [a_{(r-\alpha\circ s) \bmod n}]_{r,s=\underline{0}}^{n-e}$ are zero except for few of them given by $\sqrt{\hat{n}[0]}\sigma$ where, in our case, $\hat{n}[0] = n_1 n_2 \cdots n_{d-1}$, $n[0] = (n_1, n_2, \ldots, n_{d-1})$, and σ is any singular value of the matrix

$$\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j\right)^{1/2},$$

where C_j is an α_d-circulant matrix of dimension $n_d \times n_d$ whose expression is

$$\begin{aligned}
C_j = \left[a_{(r-\alpha\circ s) \bmod n}\right]_{r_d,s_d=0}^{n_d-1} &= \left[a_{(r_1,r_2,\ldots,r_{d-1},(r_d-\alpha_d s_d) \bmod n_d)}\right]_{r_d,s_d=0}^{n_d-1} \\
&= \left[a_{(j,(r_d-\alpha_d s_d) \bmod n_d)}\right]_{r_d,s_d=0}^{n_d-1},
\end{aligned}$$

with $(r_1, r_2, \ldots, r_{d-1}) = j$. For $j = \underline{0}, \ldots, n[0]-e$, if $C_{n_d}^{(j)}$ is the circulant matrix which has as its first column the vector $a^{(j)} = [a_{(j,0)}, a_{(j,1)}, \ldots, a_{(j,n_d-1)}]^T$ (which is the first column of the matrix C_j), $C_{n_d}^{(j)} = [a_{(j,(r-s) \bmod n_d)}]_{r,s=0}^{n_d-1} = F_{n_d} D_{n_d}^{(j)} F_{n_d}^*$, with $D_{n_d}^{(j)} = diag(\sqrt{n_d} F_{n_d}^* a^{(j)})$, then, from (30), (10), and (16), it is immediate to verify that

$$\begin{aligned}
\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j &= \sum_{j=\underline{0}}^{n[0]-e} (F_{n_d} D_{n_d}^{(j)} F_{n_d}^* Z_{n_d,\alpha_d})^* (F_{n_d} D_{n_d}^{(j)} F_{n_d}^* Z_{n_d,\alpha_d}) \\
&= \sum_{j=\underline{0}}^{n[0]-e} (F_{n_d}^* Z_{n_d,\alpha_d})^* (D_{n_d}^{(j)})^* D_{n_d}^{(j)} (F_{n_d}^* Z_{n_d,\alpha_d}) \\
&= (F_{n_d}^* Z_{n_d,\alpha_d})^* \left(\sum_{j=\underline{0}}^{n[0]-e} (D_{n_d}^{(j)})^* D_{n_d}^{(j)} \right) (F_{n_d}^* Z_{n_d,\alpha_d}).
\end{aligned}$$

Now, if we put $n_{d,\alpha} = \frac{n_d}{(n_d,\alpha_d)}$ and

$$\begin{aligned}
q_s^{(j)} &= |D_{n_d}^{(j)}|_{s,s}^2 = (D_{n_d}^{(j)})_{s,s} \cdot \overline{(D_{n_d}^{(j)})_{s,s}}, \quad s = 0, 1, \ldots, n_d - 1, \\
\Delta_l &= \begin{bmatrix} \sum_{j=\underline{0}}^{n[0]-e} q_{(l-1)n_{d,\alpha}}^{(j)} & & & \\ & \sum_{j=\underline{0}}^{n[0]-e} q_{(l-1)n_{d,\alpha}+1}^{(j)} & & \\ & & \ddots & \\ & & & \sum_{j=\underline{0}}^{n[0]-e} q_{(l-1)n_{d,\alpha}+n_{d,\alpha}-1}^{(j)} \end{bmatrix} \in \mathbb{C}^{n_{d,\alpha} \times n_{d,\alpha}},
\end{aligned}$$

for $l = 1, 2, \ldots, (n_d, \alpha_d)$, then, following the same reasoning employed for proving formula (31), we infer

$$\mathrm{Eig}\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j \right) = \frac{1}{(n_d, \alpha_d)} \mathrm{Eig}\left(J_{(n_d,\alpha_d)} \otimes \sum_{l=1}^{(n_d,\alpha_d)} \Delta_l \right),$$

where

$$\begin{aligned}
J_{(n_d,\alpha_d)} &= \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{(n_d,\alpha_d) \text{ times}}, \\
\frac{1}{(n_d, \alpha_d)} \mathrm{Eig}(J_{(n_d,\alpha_d)}) &= \{0, 1\},
\end{aligned}$$

and

$$\sum_{l=1}^{(n_d,\alpha_d)} \Delta_l = \sum_{l=1}^{(n_d,\alpha_d)} \mathrm{diag}\left(\sum_{j=\underline{0}}^{n[0]-e} q^{(j)}_{(l-1)n_{d,\alpha}+k};\ k=0,1,\ldots,n_{d,\alpha}-1\right)$$
$$= \mathrm{diag}\left(\sum_{l=1}^{(n_d,\alpha_d)}\sum_{j=\underline{0}}^{n[0]-e} q^{(j)}_{(l-1)n_{d,\alpha}+k};\ k=0,1,\ldots,n_{d,\alpha}-1\right).$$

Consequently, since $\sum\limits_{l=1}^{(n_d,\alpha_d)} \Delta_l$ is a diagonal matrix, and by exploiting basic properties of the tensor product, we find

$$\lambda_k\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j\right) = \sum_{l=1}^{(n_d,\alpha_d)}\sum_{j=\underline{0}}^{n[0]-e} q^{(j)}_{(l-1)n_{d,\alpha}+k}, \quad k=0,1,\ldots,n_{d,\alpha}-1,$$
$$\lambda_k\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j\right) = 0, \qquad k=n_{d,\alpha},\ldots,n_d-1.$$

Now, since $\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j$ is a positive semidefinite matrix, from (73) we finally have

$$\sigma_k\left(\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j\right)^{1/2}\right) = \sqrt{\sum_{l=1}^{(n_d,\alpha_d)}\sum_{j=\underline{0}}^{n[0]-e} q^{(j)}_{(l-1)n_{d,\alpha}+k}}, \quad k=0,1,\ldots,n_{d,\alpha}-1,$$
$$\sigma_k\left(\left(\sum_{j=\underline{0}}^{n[0]-e} C_j^* C_j\right)^{1/2}\right) = 0, \qquad k=n_{d,\alpha},\ldots,n_d-1.$$

7 Conclusions and future work

In this paper we have studied in detail the singular values of α-circulant matrices and we have identified the joint asymptotic distribution of α-Toeplitz sequences associated with a given integrable symbol. The generalization to the multilevel block setting has been sketched together with some intriguing relationships with the design of multigrid procedures for structured linear systems. The latter point deserves more attention and will be the subject of future researches. We also would like to study the more involved eigenvalue/eigenvector behavior both for α-circulant and α-Toeplitz structures.

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